On Birkhoff Attractors and Rotational Chaos.

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Figure: Piotr Oprocha, AGH University of Science and Technology, Kraków, Poland, photo by Andrzej Banaś

Outline

Rotation Sets for Torus Homeomorphisms

Let $h : \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphism of the 2-torus homotopic to the identity, and let $\hat{h} : \mathbb{R}^2 \to \mathbb{R}^2$ be its lift to the universal covering space $(\mathbb{R}^2, \tau)$.The *rotation set* of $\rho(\hat{h})$ is the set of accumulation points of the set

$$\left\{ \frac{\hat{h}^n(z) - z}{2\pi n} | z \in \mathbb{R}^2, n \in \mathbb{N} \right\}.$$

A similar definition exists for annulus degree 1 maps.

Preliminaries

Birkhoff attractors

- (1932) Birkhoff discovers connected attractors on the 2-torus admitting a non-unique rotation vector, for a (properly chosen) map $f = (f_1, f_2) \colon \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ which is dissipative and satisfies twist condition, that is $\sup_{x \in \mathbb{S}^1 \times \mathbb{R}} |\det(Df(x))| < 1$ and $\frac{\partial}{\partial y} f_1(x, y) > \delta > 0$

Birkhoff attractors

- LE CALVEZ, P. Propriétés des attracteurs de Birkhoff. Ergodic Theory Dyn. Syst., 8(2):241–310, 1988

Preliminaries

Strange attractors and rotational chaos

- An attractor in $\mathbb{S}^1 \times \mathbb{R}$ strange if it has two orbits with different (rational) rotation numbers. The associated dynamics is then referred to as rotational chaos.

Questions on strange attractors

- **Question** [T. Oertel-Jäger]: Can the pseudo-circle appear as a strange attractor?

Theorem (Barge&Gillette, 1991)

Suppose $h : \mathbb{A} \to \mathbb{A}$ is an orientation preserving annulus homeomorphism with an invariant cofrontier C. If the rotation number of $h|C$ is not unique then

- *C is indecomposable*
- *the set of rotation numbers contains an interval, and*
- *each rational rotation number is realized by a periodic orbit.*

Outline

Definitions

- A *continuum* is a compact and connected metric space containing at least two points.
- A continuum is *indecomposable* if it cannot be written as the union of two proper subcontinua.
- A continuum is *hereditarily indecomposable* if every subcontinuum is indecomposable.

Inverse limits

Suppose a map $f : X \to X$ is given on a metric space X. The *inverse limit space* $X_{\leftarrow} = \varprojlim\{f, X\}$ is the space given by

$$X_{\leftarrow} = \{(x_1, x_2, x_3, \ldots) \in X^{\mathbb{N}} : f(x_{i+1}) = x_i\}.$$

The topology of $X_{\leftarrow}$ is induced from the product topology of $X^{\mathbb{N}}$, with the basic open sets in $X_{\leftarrow}$ given by

$$U_{\leftarrow} = (f^{i-1}(U), f^{i-2}(U), \ldots, U, f^{-1}(U), f^{-2}(U), \ldots),$$

where U is an open subset of the $i$the factor space X. The map f is called a *bonding map*

Inverse limits

There is a natural homeomorphism $\sigma_f : X_{\leftarrow} \to X_{\leftarrow}$, called the *shift homeomorphism*, given by

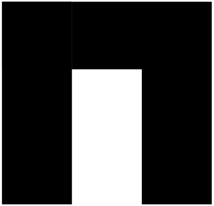
$$\sigma_f(x_1, x_2, x_3, \dots) = (f(x_1), f(x_2), f(x_3), \dots) = (f(x_1), x_1, x_2, \dots).$$

It is well known that σ_f preserves many dynamical properties of f (such as topological entropy, etc.). In particular, it is easy to see that if c is a p-periodic point of f then

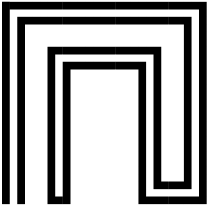
$$(f^{p-1}(c), f^{p-2}(c), \dots, c, f^{p-1}(c), f^{p-2}(c), \dots)$$

is a p-periodic point of σ_f.

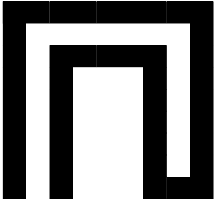
Janiszewski-Knaster Buckethandle Continuum



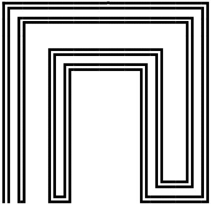
1st approximation



3rd approximation



2nd approximation



4th approximation

Circle-like cofrontiers

- A planar continuum is a *cofrontier* if it irreducibly separates the plane into exactly two components and is the boundary of each.
- A continuum is *circle-like* if it can be expressed as the inverse limit of circles.

Construction of the pseudo-circle

- (1951) R.H. Bing: pseudo-circle, a hereditarily indecomposable circle-like cofrontier

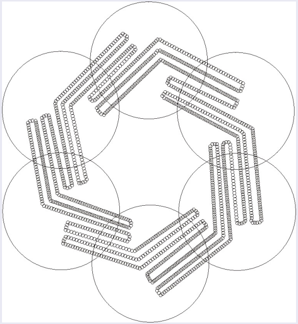


Figure: by Charatonik&Prajs&Pyrih

Construction of the pseudo-circle

- Pseudo-circle can be constructed as the intersection of a decreasing sequence of annuli A_n, where each arc A_n in $\frac{1}{n}$-crooked.

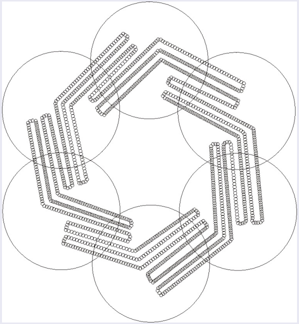
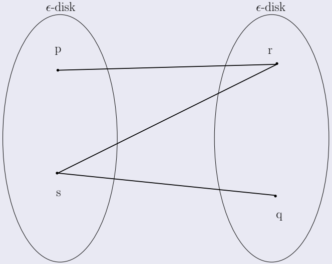


Figure: by Charatonik&Prajs&Pyrih

Construction of the pseudo-circle

- An arc is ϵ-*crooked* if for each pair of its points p and q there are points r and s between p and q on the arc such that r lies between p and s, $|p - s| < \epsilon$, and $|r - q| < \epsilon$.



Pseudo-circle

Topology of Pseudo-circle

- A space X is *homogeneous* if for every $x, y \in X$ there is a homeomorphism $h : X \to X$ such that $h(x) = y$.
- A pseudo-arc is the unique continuum homeomorphic to any subcontinuum of the pseudo-circle.
- (1948) R.H. Bing: Pseudo-arc is homogeneous.
- (1960) Fearnley, Rogers: Pseudo-circle is not homogeneous.

Topology of Pseudo-circle

- (1986) Kennedy&Rogers: Pseudo-circle is uncountably non-homogeneous.
- (2011) Sturm: Pseudo-circle is not homogeneous with respect to continuous surjections.

The pseudo-circle

Dynamics of the pseudo-circle

- (1982) Handel: pseudo-circle as a minimal set of a C^∞-smooth area-preserving planar diffeomorphism (well defined irrational rotation number).
- (1986) Kennedy&Rogers: pseudo-circle admits rational rotations.
- (1995) Kennedy& Yorke: there exist C^∞-smooth dynamical systems in dimensions greater than 2, with uncountably many minimal pseudo-circles, and any small C^1 perturbation of which manifests the same property.

The pseudo-circle

Dynamics of the pseudo-circle

- (1998) Turpin: there is an annulus diffeomorphism with the property that a countably dense set of irrational rotation numbers are represented only by pseudocircles on which the diffeomorphism acts minimally.
- (2010) Chéritat: (Herman's construction) pseudo-circle as the boundary of a Siegel disk for a holomorphic map in the complex plane.
- (2010) J.B.: for every $k > 1$ there is a $2k$-periodic orientation reversing homeomorphism of the 2-sphere with an invariant pseudo-circle.

Outline

Rotation theory

Theorem (J.B.&Oprocha)

There is a torus homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the identity (with a lift $\hat{h}$) such that h has an attracting pseudo-circle C and the rotation set of $h|C$ (with respect to $\hat{h}$) is not a unique vector.

Rotation theory

Let a piecewise linear map $\hat{g}\colon [0, 2\pi] \to \mathbb{R}$ be given by:

- $\hat{g}(0) = 2\pi/3$,
- $\hat{g}(2\pi/3) = 10\pi/3$,
- $\hat{g}(4\pi/3) = 0$,
- $\hat{g}(2\pi) = 8\pi/3$,
- and $\hat{g}$ is linear on the intervals $[0, 2\pi/3]$,$[2\pi/3, 4\pi/3]$ and $[4\pi/3, 2\pi]$.

Extend $\hat{g}$ to a map $\hat{g}\colon \mathbb{R} \to \mathbb{R}$ periodically, putting $\hat{g}(x + 2\pi) = f(x) + 2\pi$.

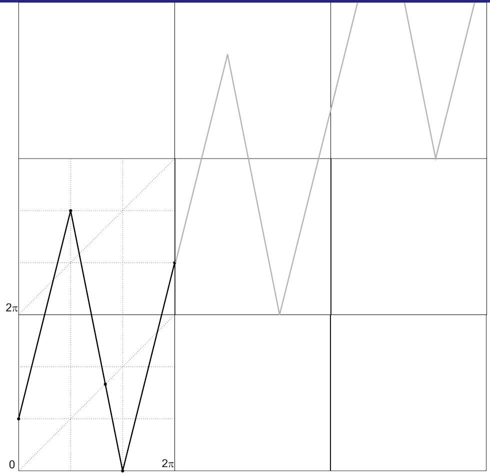


Figure: A sketch of the graph of map $\tilde{g}$.

Rotation theory

Theorem (Kościelniak, Oprocha, Tuncali, PAMS 2013)

Let $\mathbb{G}$ be a topological graph and let $\mathcal{K}$ be a (1-dimensional) triangulation of $\mathbb{G}$. For every topologically exact map $g\colon \mathbb{G} \to \mathbb{G}$ and every $\delta > 0$ there is a topologically mixing map $g_\delta\colon \mathbb{G} \to \mathbb{G}$ with the shadowing property, such that $|g - g_\delta| < \delta$, $g_\delta(x) = g(x)$ for every vertex x in $\mathcal{K}$ and the inverse limit $\varprojlim\{g_\delta, \mathbb{G}\}$ is hereditarily indecomposable.

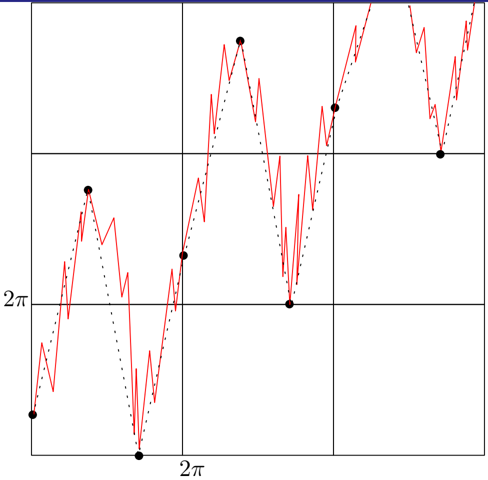


Figure: A sketch of the graph of map $\tilde{g}$.

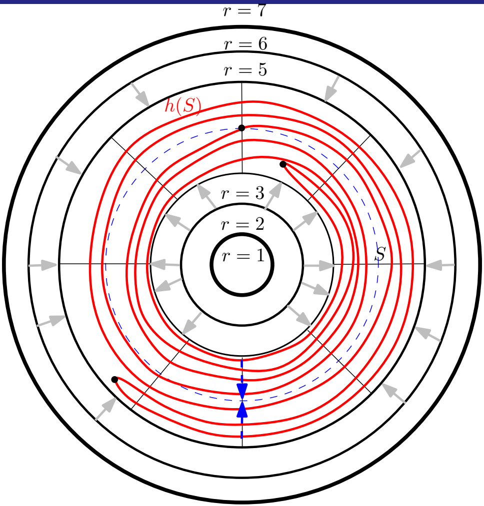


Figure: The first step of the embedding process.

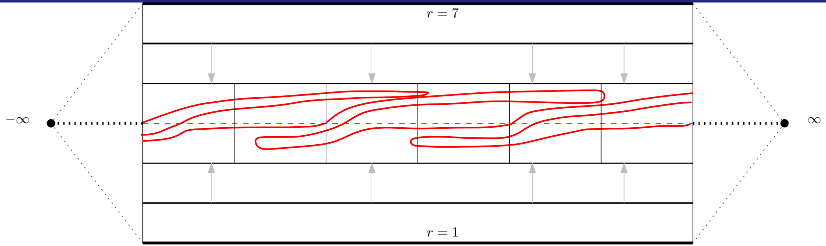


Figure: Approximation in the universal cover.

Entropy and differentiability

- **Question:** Does the pseudo-circle (or any hereditarily indecomposable continuum) admit homeomorphisms (or even maps) with finite nonzero entropy?

Entropy and differentiability

Theorem (Mouron, 2010)

Let $f : [0,1] \to [0,1]$ be a map. If the inverse limit $\varprojlim([0,1], f)$ is a pseudo-arc then the topological entropy $h(f) = h(\sigma_f) \in \{0, \infty\}$.

Theorem (J.B.&Oprocha)

Let $f : G \to G$ be a graph map. If the inverse limit $\varprojlim(G, f)$ is hereditarily indecomposable then the topological entropy $h(f) = h(\sigma_f) \in \{0, \infty\}$.

Entropy and differentiability

Lemma (M. Brown, 1960)

Let G be a topological graph. If the inverse limit $\varprojlim(G, f)$ is hereditarily indecomposable then for every $\delta > 0$ there is $n > 0$ such that each path $\omega\colon[0,1] \to G$ is (f^n, δ)-crooked.

Lemma (Llibre&Misiurewicz, 1993)

Let $f : G \to G$ be a graph map. If f has positive topological entropy (i.e. $h(f) > 0$) then there exist sequences $\{m_i\}_{i=1}^{\infty}$ and $\{k_i\}_{i=1}^{\infty}$ of positive integers such that for each n the map f^{m_n} has a k_n-horseshoe and

$$\limsup_{n\to\infty} \frac{1}{m_n} \log(k_n) = h(f).$$

Entropy and differentiability

Theorem (Ito, 1970)

Let (M, g) be a compact n-dimensional Riemannian manifold and $F : M \to M$ a C^1-diffeomorphism. Then $h(F) < \infty$, i.e. the topological entropy of F is finite.

Corollary (J.B.&Oprocha)

Let (M, g) be a compact n-dimensional Riemannian manifold and $F : M \to M$ be a homeomorphism with an invariant hereditarily indecomposable continuum X; i.e. $F(X) = X$. If $F|X$ is conjugate to a shift homeomorphism on a graph inverse limit $\varprojlim(G, f)$ then either $h(F) = 0$ or F is non-differentiable.

Open questions

Problem 1: *Are there uncountably many topologically distinct connected attractors in* $\mathbb{T}^2$ *that admit a non-unique rotation vector?*
Problem 2: *Characterize the inverse limits* $\varprojlim\{\mathbb{S}^1, f\}$ *for degree* 1 *circle maps* f*, that admit two distinct rotation numbers. What conditions on such* $f : \mathbb{S}^1 \to \mathbb{S}^1$ *and* $g : \mathbb{S}^1 \to \mathbb{S}^1$ *assure that* $\varprojlim\{\mathbb{S}^1, f\}$ *and* $\varprojlim\{\mathbb{S}^1, g\}$ *are topologically distinct?*
Problem 3: *Is there a hereditarily indecomposable attractor (like "pseudo-figure-eight") on the 2-torus with a 2-dimensional rotation set* R *(i.e.* $int(R) \neq \emptyset$*)?*

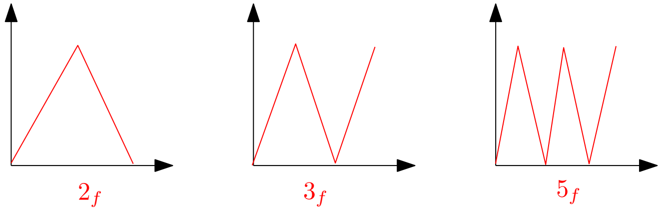
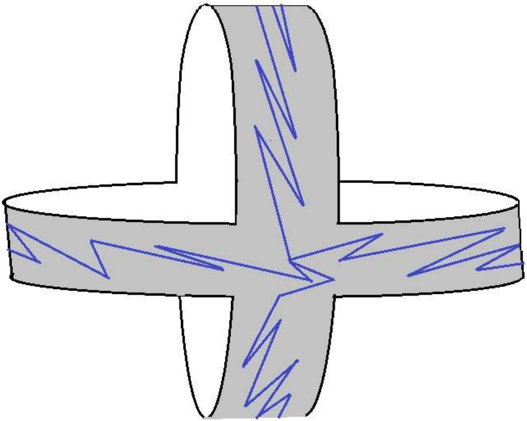


Figure: Generating buckethandle continua.

Theorem (Watkins, 1982)

The buckethandle continua $\varprojlim\{[0,1], N_f\}$ *and* $\varprojlim\{[0,1], M_f\}$ *are homeomorphic if and only if N and M have the same prime factors.*



Thanksgiving

Thank You for Your Attention