New rotation sets in a family of toral homeomorphisms

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Outline

- The rotation sets of torus homeomorphisms: general definitions, results, and questions.
- Description of the rotation sets in our family f_ν.
- Steps in constructing and analyzing the family:
  - A family of maps of the figure-eight viewed as the spine of the punctured two-dimensional torus.
  - Rotation set of this family is carried by an embedded (inverse limit of a simple tower over a) beta-shift
  - Analyzing the digit frequency sets of beta-shifts (Hall's talk)
  - Unwrapping of the figure eight maps to a family of torus homeomorphisms using the inverse limit.
- There is an analogous construction for higher dimensional tori yielding an similar theorem.

The rotation sets of torus homeomorphisms: definitions, questions, and results

Definitions of rotation vector and set

- Let $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphism of the two-dimensional torus isotopic to the identity.
- Fix a lift to the universal cover $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$. It defines a displacement cocycle $D : \mathbb{T}^2 \to \mathbb{R}^2$ via $D(z) := \tilde{f}(\tilde{z}) - \tilde{z}$ where $\tilde{z}$ is any lift of z.
- The displacement after n iterates is the dynamical cocycle

$$D(z,n) := D(z) + \cdots + D(f^{n-1}(z)) = \tilde{f}^n(\tilde{z}) - \tilde{z})$$

- The pointwise rotation vector is the average displacement.

$$\rho_p(z) = \lim_{n\to\infty} \frac{D(z,n)}{n} = \lim_{n\to\infty} \frac{\tilde{f}^n(\tilde{z}) - \tilde{x}}{n},$$

if the limit exists (note actual Birkhoff limit).

Definitions of rotation vector and set

- The pointwise rotation set of f is

$$\rho_p(f) = \{\rho_p(z) \colon z \in \mathbb{T}^2\}$$

- The pointwise rotation set is the most natural definition but it is difficult to understand directly.
- Misiurewicz & Ziemian proposed the now standard definition of the rotation set as

$$\rho(f) = \{\mathbf{v} \in \mathbb{R}^2 : \frac{D(z_i, n_i)}{n_i} \to \mathbf{v} \text{ with } z_i \in \mathbb{T}^2, n_i \to \infty\}$$

- Obviously $\rho_p(f) \subset \rho(f)$.

Basic questions

- Question 1: Shapes What are the possible geometric shapes of $\rho(f)$ for torus homeomorphisms f isotopic to the identity?
- Question 2: Dynamics How much does the rotation set tell you about the dynamics of its homeomorphism. Is $\rho_p(f) = \rho(f)$? For each $\mathbf{v} \in \rho(f)$ is there a nice compact invariant set $X_{\mathbf{v}}$ with $\rho(X_{\mathbf{v}}) = \mathbf{v}$ or an ergodic invariant measure ν with $\mathbf{v} = \int D \, d\mu$
- Question 3: Bifurcations How does the rotation set change in parameterized families and what is the generic shape?

Some terminology

- $\mathrm{Homeo}_0(\mathbb{T}^2)$ is all homeomorphisms of the two-torus that are isotopic to the identity.
- A rational polygon in the plane is a convex region with interior that has finitely many extreme points each of which is contained in $\mathbb{Q}^2$.
- A extreme point p of a planar convex body C is called a vertex if $\mathrm{Bd}(C)$ is locally isometric to a polygon vertex, or equivalently, if p is isolated $\mathrm{Ex}(C)$, the set of extreme points of C.
- A vector $\mathbf{v} \in \mathbb{R}^2$ is irrational if $\mathbf{v} \notin \mathbb{Q}^2$, partially irrational if $\mathbf{v} \cdot \mathbf{n} = 0$ for some nonzero $\mathbf{n} \in \mathbb{Z}^2$ and totally irrational if it is irrational and not partially irrational. Thus $\mathbf{v}$ is totally irrational iff $z \mapsto z + \mathbf{v}$ is minimal on $\mathbb{T}^2$.

What's Known – Question 1: Shapes

- (Misiurewicz & Ziemian) $\rho(h)$ is always a compact, convex set in $\mathbb{R}^2$, and thus is either a point, an interval or has interior.
- Franks–Misiurewicz conjecture (?): If $\rho(h)$ is a nontrivial segment then either it has a rational endpoint or else it contains infinitely many rational points.
- (Kwapisz) Any rational polygon is $\rho(h)$ for some $h \in \mathrm{Diff}_0^\infty(\mathbb{T}^2)$.
- (Kwapisz) There exist $h \in \mathrm{Diff}_0^1(\mathbb{T}^2)$ so that $\rho(h)$ has countably infinite many rational vertices with two limiting extreme points which are partially irrational.

What's Known – Question 2: Dynamics

- Much recent work on case where $\rho(h)$ is a point or interval, and sometimes with area-preserving. Main focus here $\mathrm{Int}(\rho(h)) \neq \emptyset$.
- (Llibre-MacKay) $\mathrm{Int}(\rho(h)) \neq \emptyset$ implies $h_{\mathrm{top}}(h) > 0$.
- (Franks) For each $\mathbf{p}/q \in \mathrm{Int}(\rho(h))$ there exists a $\mathbf{p}/q$-periodic point.
- (M.& Z.) For any $\mathbf{v} \in \mathrm{Int}(\rho(h))$ there is an invariant minimal set $X_{\mathbf{v}}$ with $\rho(X_{\mathbf{v}}) = \mathbf{v}$. For any $\mathbf{v} \in \mathrm{Int}(\rho(h)) \cup \mathrm{Ex}(\rho(h))$ there is an ergodic invariant measure μ with $\int D\, d\mu = \mathbf{v}$. There are h which have points $\mathbf{v} \in \mathrm{Bd}(\rho(h))$ for which there are no compact invariant sets $X_{\mathbf{v}}$ with $\rho(X_{\mathbf{v}}) = \mathbf{v}$.
- Putting these together, when $\rho(h)$ has interior, $\rho(h) = \mathrm{Cl}(\rho_p(h))$ with the closure just perhaps adding boundary points.

What's Known – Question 3: Bifurcations

- Definition: The collection of compact, convex subsets of the plane is $\mathcal{H}(\mathbb{R}^2)$ and is given the Hausdorff topology and partially ordered by inclusion.
- (Misiurewicz & Ziemian) If f_ν is a continuous family of homeomorphisms and $\rho(f_0)$ has interior, then $\nu \mapsto \rho(f_\mu) \in \mathcal{H}(\mathbb{R}^2)$ is continuous in a neighborhood of $\nu = 0$.
- (Passeggi) The collection of $h \in \mathrm{Homeo}_0(\mathbb{T}^2)$ with $\rho(h)$ a rational polygon contains a C^0-open, dense set (Note: The cases $\rho(h)$ is a point or segment can be included in this set).
- (Zanata) If $\rho(h)$ has an irrational extreme point $\mathbf{v}$, then there exists a homeomorphism f arbitrarily C^0-close to h so that $\rho(f) \neq \rho(h)$ and $\rho(f) \cap \rho(h)^c \neq \emptyset$.

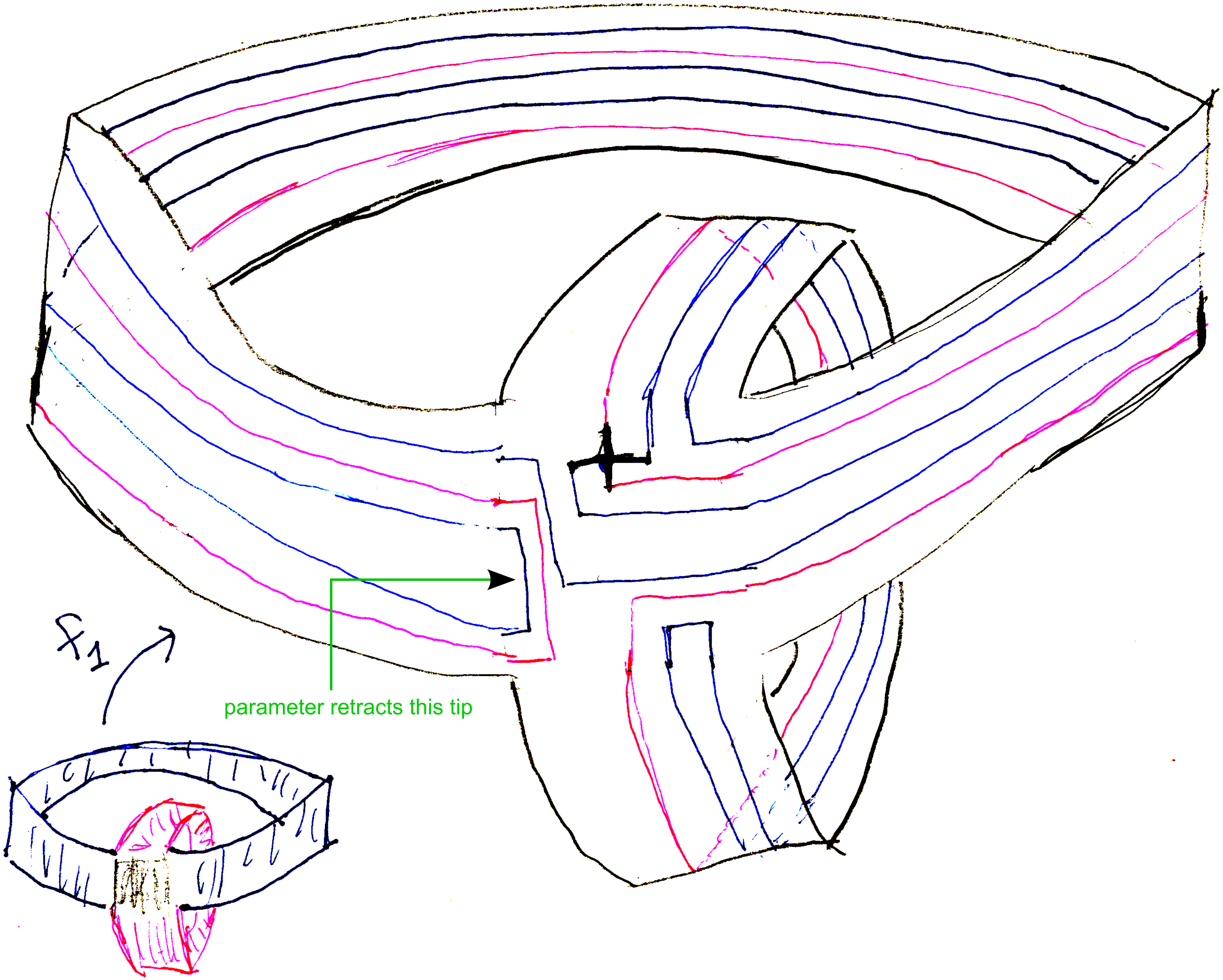
The family of torus homeomorphisms

Motivation

- For non-empty interior do the known shapes of rotation sets coupled with their Hausdorff continuity give a complete picture of the behaviour of rotation sets in a family?
- The goal was to construct a family in which all the rotation sets and their changes could be described explicitly.
- The family exhibits new phenonena and can be used to test and formulate conjectures.

Informal description

- The family of torus homeomorphisms is denoted f_ν with $\nu \in [0,1]$, and we write $\rho(\nu) := \rho_{MZ}(f_\nu)$.
- Roughly, the rotations sets behave like the rotation numbers of a family of circle homeomorphisms.
  - The bifurcations of the rotation set take place on a Cantor set $\mathcal{B}$.
  - On the closure of the complementary gaps of $\mathcal{B}$ the rotation set mode locks as a "rational structure", namely, a rational polygon.
  - On buried points of $\mathcal{B}$ the rotation set has "irrational structure" with either one or two irrational limit extreme points.
- Movie



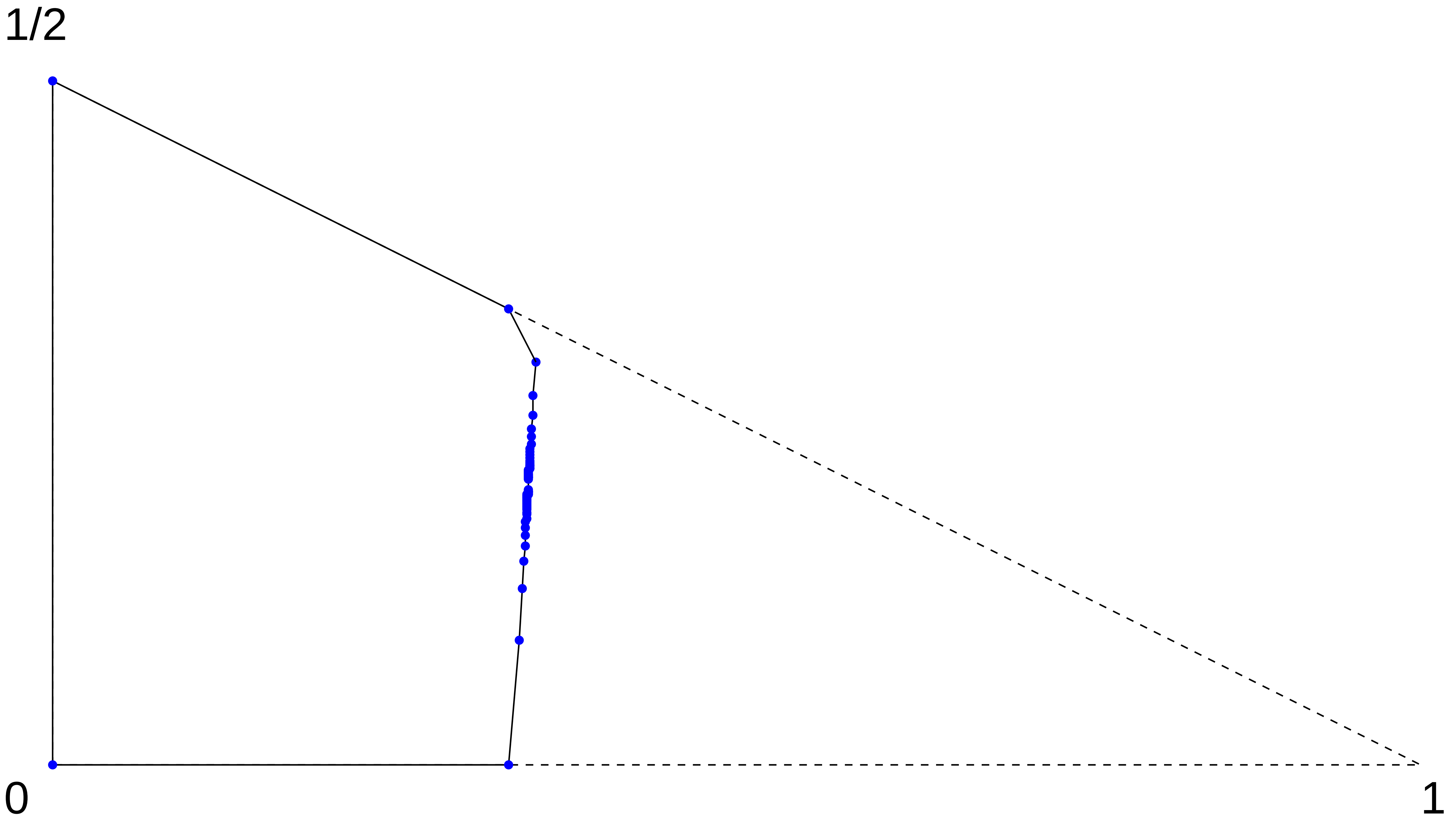
parameter retracts this tip
f_1

Theorem on Shapes and Bifurcations

- A parameter ν_0 is called a bifurcation point if there are ν arbitrarily close to ν_0 with $\rho(\nu) \neq \rho(\nu_0)$.
- The bifurcation locus of the family, $\mathcal{B} \subset [0,1]$, is a zero measure Cantor set.
- For all ν the point-wise and MZ-rotation sets are the same, $\rho_p(f_\nu) = \rho_{MZ}(f_\nu)$, and have interior.
- $\nu \mapsto \rho(\nu) \in \mathcal{H}(\mathbb{R}^2)$ is continuous (MZ) and nondecreasing.
- The parameter space admits a disjoint decomposition $[0,1] = P_1 \sqcup P_2 \sqcup P_3$.
  - $P_1 = \cup[\ell_n, r_n]$ with $\ell_n, r_n \in \mathcal{B}$ and P_1 is full measure in $[0,1]$, so P_1 is the generic case in the parameter.
  - $P_2 \sqcup P_3 \subset \mathcal{B}$ and consists of buried points in the Cantor set $\mathcal{B}$. Each of P_2 and P_3 is an uncountable set which is dense in $\mathcal{B}$.

Theorem on Shapes and Bifurcations

- If $\nu \in P_1 = \cup[\ell_n, r_n]$, then $\rho(\nu)$ is a rational polygon which is constant for $\nu \in [\ell_n, r_n]$.
- If $\nu \in P_2$, then $\mathrm{Ex}(\rho(\nu))$ consists of countably many rational vertices and one limit, irrational extreme point.
- If $\nu \in P_3$, then $\mathrm{Ex}(\rho(\nu))$ consists of countably many rational vertices and two limit, irrational extreme points. There is an exceptional interval between these two extreme points that is on $\mathrm{Bd}(\rho(\nu))$
- The rational vertices of each $\rho(\nu)$ can be algorithmically determined.
- $\nu \mapsto \mathrm{Ex}(\rho(\nu)) \in \mathcal{H}(\mathbb{R}^2)$ is discontinuous for $t \in P_3$ and continuous elswhere (the Tal-Zanata property).



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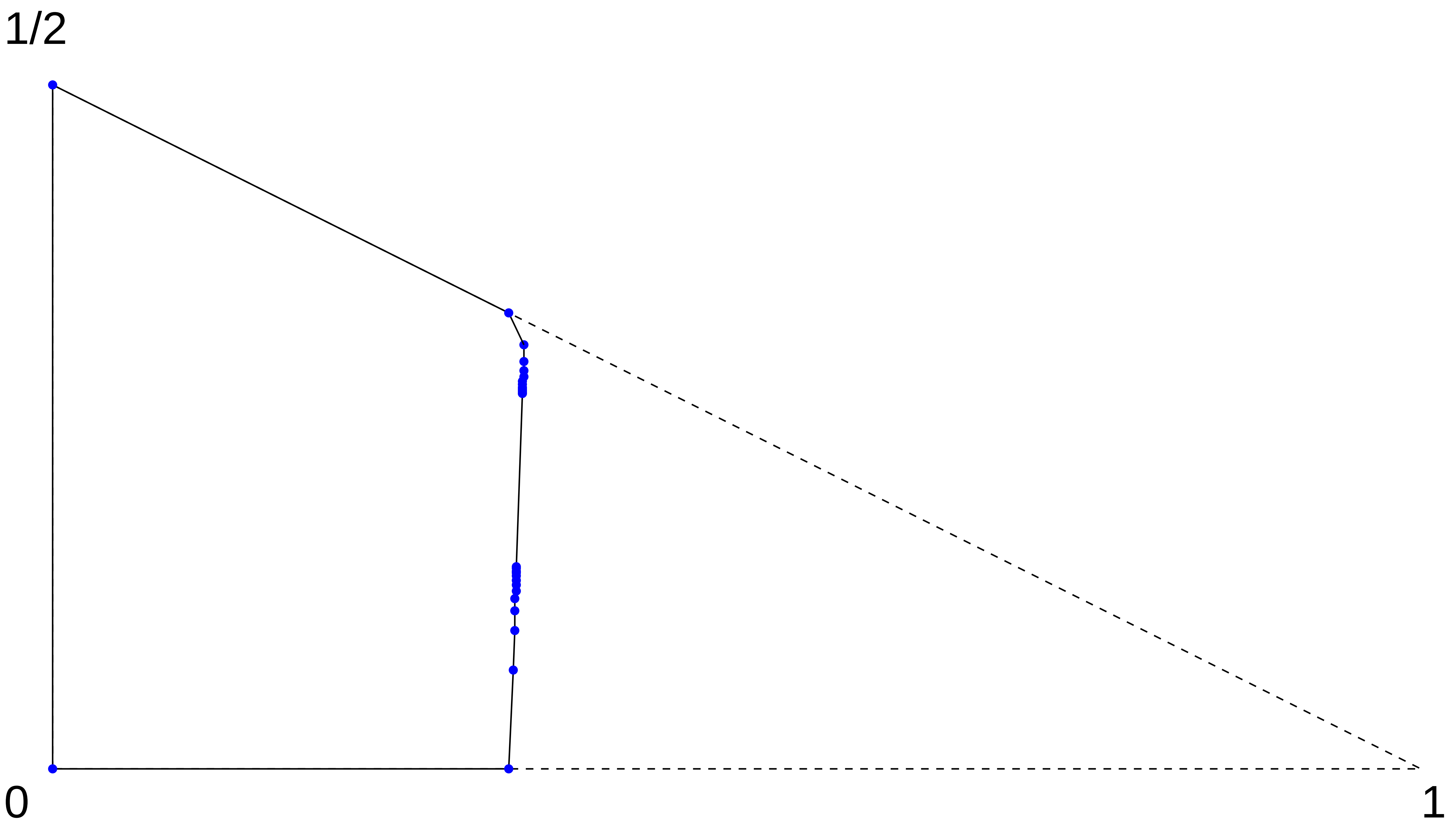
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Theorem on Shapes and Bifurcations

- The collection of all rotation sets

$$\{\rho(f_\nu) \colon \nu \in [0,1]\} \subset \mathcal{H}(\mathbb{R}^2)$$

  with the Hausdorff topology is topologically a closed interval $\mathcal{I}$.

- The projection $\pi : [0,1] \to \mathcal{I}$ via $\nu \mapsto \rho(\nu) \in \mathcal{H}(\mathbb{R}^2)$ simply collapses the intervals of P_1 to points.
- Each of $\pi(P_1), \pi(P_2)$, and $\pi(P_2)$ is dense in $\mathcal{I}$.
- Let $P_2' \subset P_2$ consists of all those ν for which the limit extreme point of $\rho(\nu)$ is totally irrational, then $\pi(P_2')$ contains a dense, G_δ-set in the interval $\mathcal{I}$. Thus amongst all the rotation sets in the family given the Hausdorff topology the generic case is to have a single, totally irrational limit extreme point.

Dynamics associated with the rotation sets

- Given $\mathbf{v} \in \rho(\nu)$ the nicest dynamical representative of $\mathbf{v}$ would be (semi)conjugate to an invariant set of rigid translation on the torus by $\mathbf{v}$. The next definitions isolate various properties posssed by this nicest representative.
- An invariant set $Z \subset \mathbb{T}^2$ is called a $\mathbf{v}$-set if $\rho_p(Z) = \mathbf{v}$, i.e. every $z \in Z$ has pointwise rotation vector $\mathbf{v}$.
- A $\mathbf{v}$-set Z is said to have bounded deviation if there exists an M so that

$$\|D(z, n) - n\mathbf{v}\| < M \tag{1}$$

for all $n \in \mathbb{N}$ and $z \in Z$.

Dynamics associated with the rotation sets

- An invariant measure μ whose support is a $(\int D \, d\mu)$-set is called directional and otherwise is called lost.
- Note that f has a directional ergodic measure μ if and only it has a $(\int D \, d\mu)$-minimal set.
- Recall that Oxtoby's Theorem says that (Z, g) is uniquely ergodic iff for every continuous observable the Birkhoff average at ecery point comverges.
- Thus if $X \subset \mathbb{T}^2$ is uniquely ergodic unique invariant measure μ, then its unique invariat measure is directional, and directional is in this way an analog of uniquely ergodic focused just on the displacement observable.

Dynamics associated with the rotation sets

- Using this language the best dynamical representative of some $\mathbf{v} \in \rho(\nu)$ would be a uniquely ergodic, $\mathbf{v}$-mimimal set with bounded deviation.
- The unique ergdicity is hard to obtain and we adapt a slightly weaker definition of a $\mathbf{v}$-good set as an invariant set that is
  1. a $\mathbf{v}$-set,
  2. a minimal set,
  3. of bounded deviation,
  4. and the support of a directional measure.
- Theorem (MZ): For all $\mathbf{v} \in \mathrm{Int}(\rho(h))$ there exists a $\mathbf{v}$-good set $Z_{\mathbf{v}}$.
- By Franks' Theorem when $\mathbf{v}$ is rational, $Z_{\mathbf{v}}$ may be chosen to be a periodic point.

Dynamics associated with the rotation sets

- So any remaining questions involve $\mathbf{v}$-good sets for $\mathbf{v} \in \mathrm{Bd}(\rho(h))$.
- Theorem: For all $\nu \in [0,1]$ and for all $\mathbf{v} \in \mathrm{Bd}(\rho(\nu)))$ there exists a $\mathbf{v}$-good set except for the following two cases:

Exception 1

- When $\nu \in P_2$, the rotation set $\rho(\nu)$ has a single irrational extreme point which we denote $\mathbf{w}_\nu$.
- There always exists a uniquely ergodic, $\mathbf{w}_\nu$-minimal set $Z_\mathbf{w}$, but there are examples in the family where this minimal set is never of bounded deviation.
- Specifically, for some $z \in Z_\mathbf{w}$ and subsequence $n_i \to \infty$,

$$\|D(z, n_i) - n_i \mathbf{v}\| \sim {n_i}^\eta \text{ with } 0 < \eta < 1.$$

- So $D(z, n_i)/n_i \to \mathbf{v}$ at a rate of ${n_i}^{\eta - 1}$.

Exception 2

- When $\nu \in P_3$, the rotation set $\rho(\nu)$ has a two irrational extreme points q_1 and q_2 which are the endpoints of an interval $J_\nu \subset \mathrm{Bd}(\rho(\nu))$.
- There exists a minimal set X with $\rho_{MZ}(X) = J_\nu$.
- This minimal set supports exactly two ergodic invariant measures μ_i with $\int D \, d\mu_i = q_i$.
- X is unique in the sense that any minimal set X' with $\rho_{MZ}(X') \cap J_\nu \neq \emptyset$ has $\rho_{MZ}(X) = J_\nu$ and is conjugate to X.
- Thus for $\mathbf{v} \in J_\nu$, there is no $\mathbf{v}$-set and the only ergodic invariant measures representing the limit extreme points q_1 and q_2 are lost.
- Note that for $\mathbf{v} \in J_\nu$ there is a point z with $\rho_p(z) = \mathbf{v}$, but that point doesn't have bounded deviation as that would imply that its ω-limit set is a $\mathbf{v}$-set.

Zanata's condition

- If h is $C^{1+\epsilon}$ and μ is an ergodic measure with $\rho(\mu)$ an extreme point with multiple supporting lines, then μ is directional and the support has bounded deviation.
- If h is $C^{1+\epsilon}$ and μ is an ergodic measure with $\rho(\mu)$ an extreme point with one supporting line which does not intersect $\rho(h)$ in a nontrivial segment, then μ is directional but no conclusion about bounded deviation.
- Our family is C^0, but the examples do not contradict these results since the other hypothesis do not hold.
  - In exception 1, the limit extreme point has a unique supporting line.
  - In exception 2, the supporting line intersects the rotation set in a nontrivial interval.

Questions prompted by family

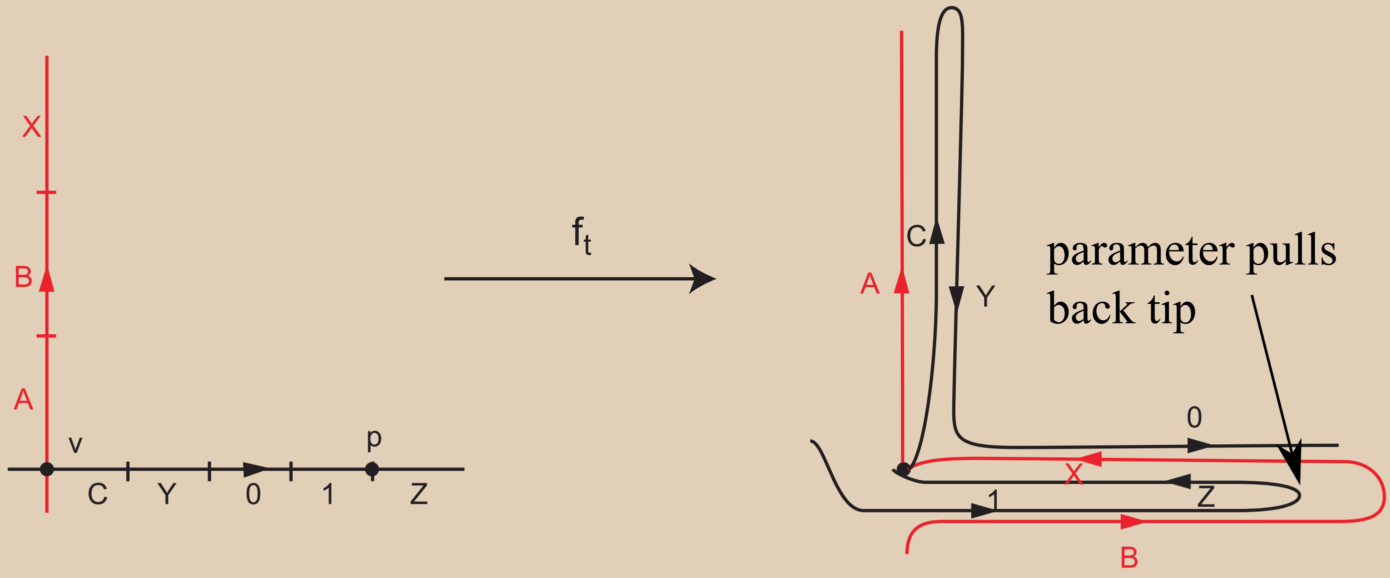
- In this family:
  - A point is a vertex iff it is a rational extreme point and is a limiting extreme point iff it is an irrational extreme point.
  - For every point $\mathbf{v} \in \rho(f_t)$ (including points on the boundary) there is a point x with $\rho(x, f_t) = \mathbf{v}$ and so $\rho_p(f_\nu) = \rho(f_\nu)$.

  Are these always true?
- If g_t is a continuous family with $\rho(g_a) \neq \rho(g_b)$ is the set of extreme points of $\rho(g_t)$ always discontinuous at some points t? (the Tal-Zanata property)
- The crucial techniques for constructing and analyzing this family are all C^0, what happens with more smoothness?

From Figure-Eight Maps to Beta-shifts to Torus Homeomorphisms.

The family of maps of the figure-eight

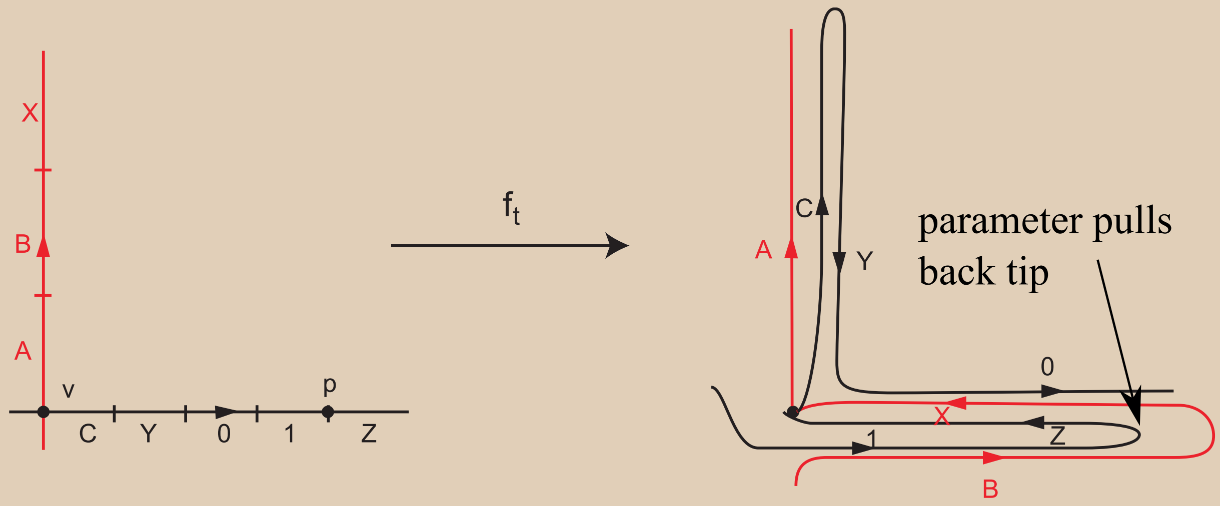
- The figure shows a family g_ν of maps of the wedge of two circles or figure-eight E.
- The maps are homotopic to the identity and all fix the junction point P. The parameter ν moves the tip.
- Identify points mod one in both directions in figure:



Rotation vectors for the family g_ν

- Rotation about each circle gives a component of the two-dimensional rotation vector of a point under g_ν (formalized using the universal Abelian cover of the figure eight).
- The rotation set is denoted $\rho(g_\nu)$.

Reduction to a beta-shift



- For each ν, dynamics of g_ν is coded by a subshift of $\{A, B, X, C, Y, 0, 1, Z\}^{\mathbb{N}}$ obtained by pruning away the sequences for points that land beyond the tip.
- Since we just want to compute the rotation sets we find a smaller subshift that has the same rotation set and is easier to work with. For example, intervals X, Y, Z were the orientation is reversed are not needed.

Reduction to a beta-shift

- The state A also turns out to be not needed and the states B and C only occur as the pair CB, so we call that symbol W.
- We are left with a subshift of $\{W,0,1\}^{\mathbb{N}}$. Using the induced order from $W<0<1$ this subshift is

$$\{\underline{s} \in \{W,0,1\}^{\mathbb{N}} : \sigma^k(\underline{s}) \leq \underline{t} \text{ for all } k \in \mathbb{N}\}$$

  for the appropriate "kneading sequence" $\underline{t}$.
- Such subshifts have been extensively studied as beta-shifts. They are Parry's symbolic description of Renyi's β-expansions of real numbers.
- Thus to understand the rotation set of g_ν we need to understand the asymptotic averages of different symbols in a beta-shift (Toby Hall's talk)

Extending to a family of torus homeomorphisms

- The last step is to use the family of maps of the figure-eight to get a family on the torus.
- Fattening up to a fibered neighborhood doesn't give enough control over the rotation sets.
- A construction of Barge & Martin using a theorem of M. Brown allows us for each ν to construct a homeomorphism of a neighborhood of $E \subset \mathbb{T}^2$ which has the inverse limit $\varprojlim(E, g_\nu)$ as an attractor. Adding a single repelling fixed point gives a homeomorphism f_ν of the torus.
- The construction gives you enough control to get the same rotation sets, $\rho(f_\nu) = \rho(g_\nu)$.
- More work is required to get the construction continuous in the parameter as needed to understand the family of rotation sets.