Fixed point indices via Conley theory

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Joint work with:
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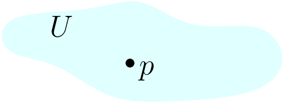
(Local) fixed-point index

- $f : \mathbb{R}^d \to \mathbb{R}^d$.
- $f(p) = p$ and p isolated (in $\mathrm{Fix}(f)$).

Fixed point index of f at p, $i(f, p) \in \mathbb{Z}$, measures the multiplicity of p as a fixed point.



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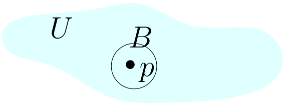
$$B \cap Fix(f) = \{p\}$$



$$g : \partial B \to S^{d-1}$$

$$x \mapsto g(x) := \frac{x - f(x)}{\|x - f(x)\|}$$

Definition (Fixed point index at p)

$$i(f,p) = deg(g) \in \mathbb{Z}$$

Remark: If f is C^1 and $I - Df$ is regular at p then $i(f,p) = \pm 1$.

Results about the integer sequence $\{i(f^n, p)\}_{n\geq 1}$

Dold's congruences

$I = \{i(f^n, p)\}_{n\geq 1}$ satisfies $$I = \sum_{k\geq 1} a_k \sigma^k$$

with $a_k \in \mathbb{Z}$ and $\sigma^k = (0, 0, \ldots, k, 0, 0, \ldots, k, 0, \ldots)$.

C^1 maps (Shub – Sullivan)

$\{i(f^n, p)\}_{n\geq 1}$ is bounded $\Leftrightarrow$ is periodic.

Planar **homeomorphisms**

- (Nikishin, Simon, Pelikan–Slaminka, Bonino) f preserves a Borel measure $\Rightarrow$ $i(f, p) \leq 1$.
- (Brown, Le Calvez, Le Roux) f orientation–preserving:
$$i(f^n, p) = \begin{cases} 1 & \text{otherwise} \\ s & \text{if } n \equiv 0 (\text{mod } q) \end{cases} \quad \text{for some } s \in \mathbb{Z}, q \geq 1.$$
- (Bonino) f orientation–reversing $\Rightarrow i(f, p) = -1, 0, 1$.

Isolation of the fixed point

Definition (Locally maximal fixed point)

$f(p) = p$ is said to be **isolated as an invariant set** or **locally maximal** if there exists W nbd of p such that the unique **full** orbit contained in W is $\{p\}$. W is called an *isolating neighborhood*.

Definition ("Dynamically adapted" neighborhoods)

B is an **isolating block** of a locally maximal fixed point p if:

- B is an isolating neighborhood.
- B has no interior "tangencies", i.e., $x, f(x), f^2(x) \in B \Rightarrow f(x) \notin \partial B$.

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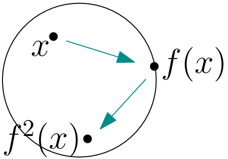

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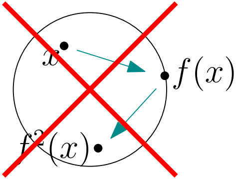

Any locally maximal p has a **basis** of neighborhoods composed of **isolating blocks**.

Index of locally maximal fixed points in $\mathbb{R}^2$

Theorem (Le Calvez – Yoccoz, Ruiz del Portal – Salazar)

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and p a locally maximal fixed point of f, neither a sink nor a source. Then,

a) *if $f \in \mathrm{Homeo}^+(\mathbb{R}^2)$*

$$i(f^n, p) = \begin{cases} 1 & \text{otherwise} \\ 1 - qr & \text{if } n \equiv 0 (\text{mod } q) \end{cases} \quad \text{for some } r, q \geq 1.$$

b) *if $f \in \mathrm{Homeo}^-(\mathbb{R}^2)$*

$$i(f^n, p) = \begin{cases} a & \text{odd } n \\ a - 2r & \text{even } n \end{cases} \quad \text{for some } a \in \{-1, 0, 1\}, r \geq 1.$$

In particular, always $i(f^n, p) \leq 1$ and $\{i(f^n, p)\}_{n \geq 1}$ is periodic.

Corollary (Brouwer, Handel, Le Calvez–Yoccoz)

$\nexists$ *minimal homeomorphisms in $S^2 \setminus \{p_1, \dots, p_r\}$.*

What result is expected if f only continuous?

Let $\Phi : S^1 \to S^1$ and J a finite union of periodic orbits of Φ. Denote $\varphi := \Phi_{|J}$.

Description of $\#\mathrm{Fix}(\varphi^n)$:

$\#\mathrm{Fix}(\varphi^n) = \sum_{q|n} q \cdot \#\{\text{orbits of period } q\}$

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- $\Phi \in \mathrm{Homeo}^-(S^1) \Rightarrow$ up to 2 fixed points and 2–periodic orbits

$$\#\mathrm{Fix}(\varphi^n) = \begin{cases} b & \text{odd } n \\ b + 2r & \text{even } n \end{cases} \quad \text{for some } b \in \{0, 1, 2\}, r \geq 1.$$

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a) *if $f \in \mathrm{Homeo}^+(\mathbb{R}^2)$ $\Rightarrow \Phi \in \mathrm{Homeo}^+(S^1)$ and*

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Index of locally maximal fixed points in $\mathbb{R}^2$

Theorem

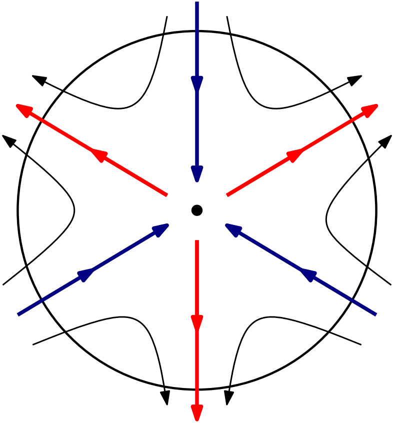
Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a **continuous** *map and p a locally maximal fixed point of f, neither a sink nor a source. Then, $\exists J$ finite and $\varphi : J \to J$ (no restriction over the possible periods of φ, J) such that*

$$i(f^n, p) = 1 - \#\mathrm{Fix}(\varphi^n).$$

In particular, always $i(f^n, p) \leq 1$ and $\{i(f^n, p)\}_{n \geq 1}$ is periodic.

Example

Let $f = R_{2\pi/3} \circ \Psi^1$, Ψ^1 is time–1 map of the flow depicted below.
Local unstable set, Local stable set



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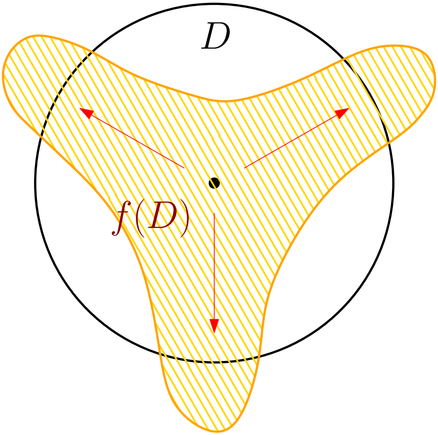

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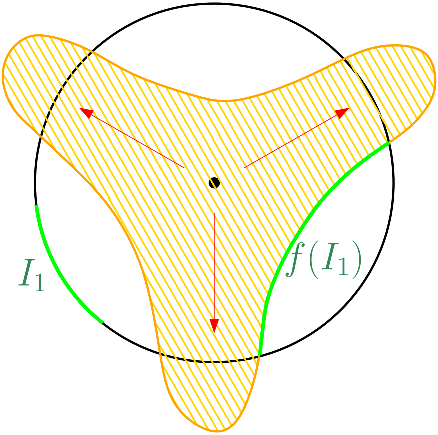

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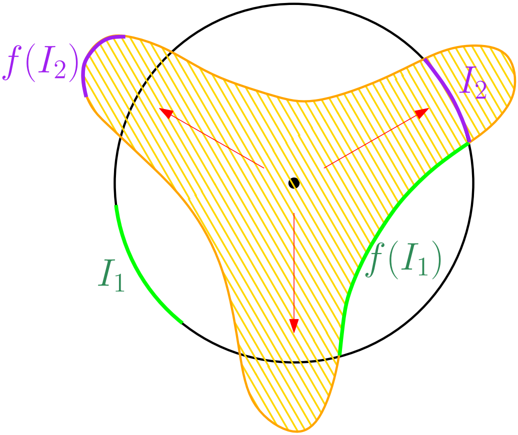

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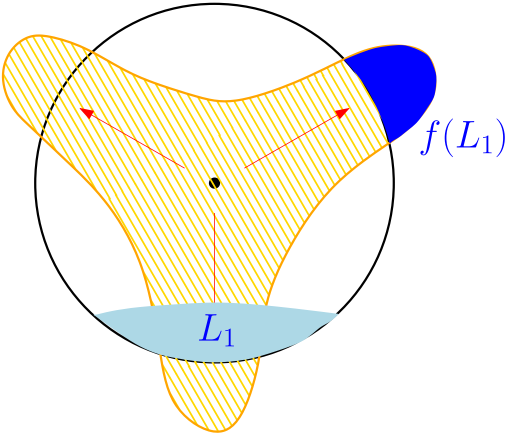

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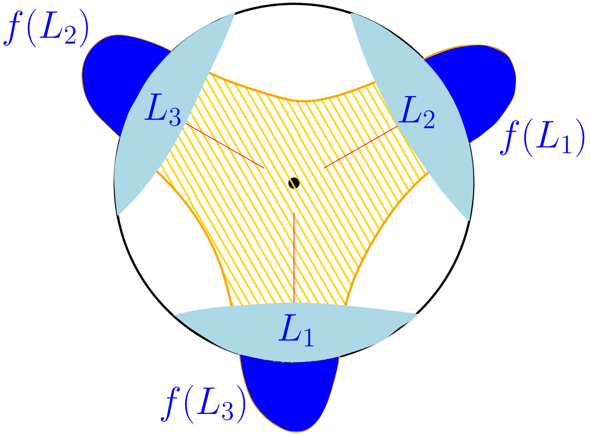
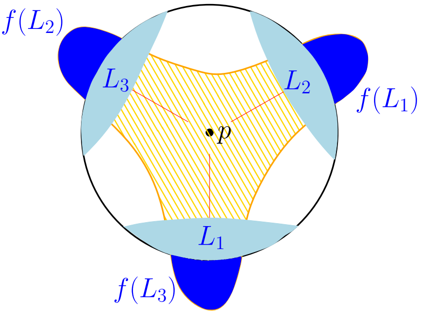
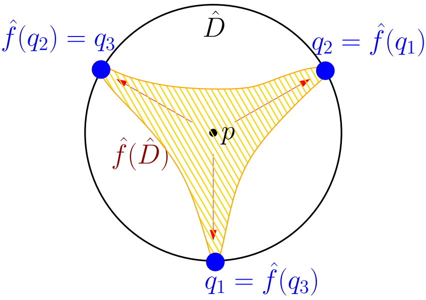


Example

$f : D \to f(D)$



$\hat{f} : \hat{D} \to \hat{D}$



$\{q_1, q_2, q_3\}$ attracting 3–periodic orbit of $\hat{f}$

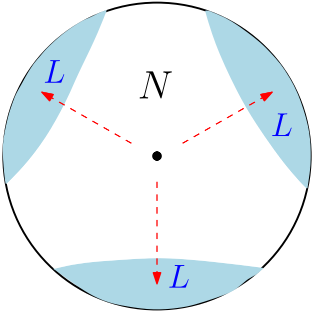
$$1 \xrightarrow{\varphi} 2 \xrightarrow{\varphi} 3 \xrightarrow{\varphi} 1$$

$$i(f^n, p) = \begin{cases} 1 & \text{otherwise} \\ 1 - 3 & \text{if } n \equiv 0 (\text{mod } 3) \end{cases}$$

Link with (discrete) Conley index

$f : \mathbb{R}^d \to \mathbb{R}^d$ **continuous** map, p **locally maximal** fixed point

N isolating block of $\{p\}$, $L :=$ small nbd of $f^{-1}(N^c) \cap N$



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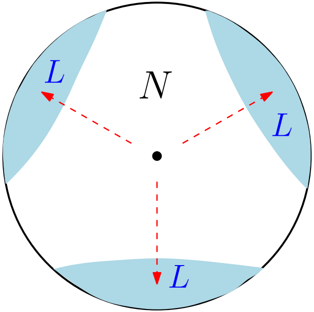
Define induced map $\boxed{\bar{f} : N/L \to N/L}$, the "fat" point $[L]$ is an attracting fixed point for $\bar{f}$.



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Theorem (Robbin–Salamon, Mrozek, Szymczak, Franks–Richeson)

The non–nilpotent part of $\bar{f}_* : H_*(N/L) \to H_*(N/L)$ is independent of (N, L). The invariant is called (discrete homological) Conley index, denoted $h(f, p)$.

$$h_r(f, p) \ni \bar{f}_{*,r} : H_r(N/L) \to H_r(N/L).$$

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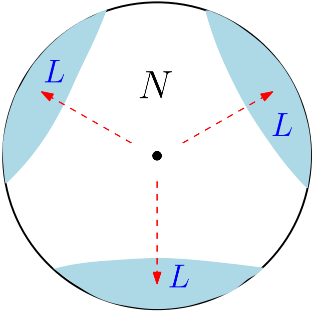

Where is $\{q_1, q_2, q_3\}$ in this description?

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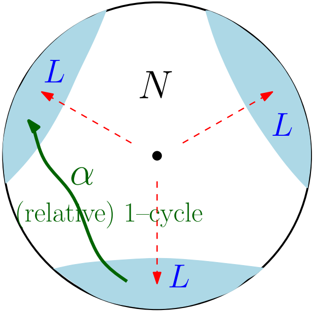

Where is $\{q_1, q_2, q_3\}$ in this description?

It is encoded in

$$\bar{f}_{*,1} : H_1(N/L) \to H_1(N/L),$$

i.e., in the first Conley index.

$\alpha \in H_1(N/L)$ is 3–periodic under $\bar{f}_{*,1}$.

Local Lefschetz formula

Applying Lefschetz Theorem to the induced map $\bar{f} : N/L \to N/L$ yields:

$$i(\bar{f}, p) + i(\bar{f}, [L]) = \Lambda(\bar{f}) = 1 + \sum_{r \geq 0} (-1)^r \mathrm{trace}(\bar{f}_{*,r})$$

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$\mathrm{trace}(\bar{f}_{*,r})$ only depends on the non-nilpotent part of $\bar{f}_{*,r} \Rightarrow$
$\Rightarrow$ it only depends on the Conley index $h_r(f,p)$.

Theorem (Mrozek)

$$i(f,p) = \sum_{r \geq 0} (-1)^r \mathrm{trace}(\bar{f}_{*,r}) = \sum_{r \geq 0} (-1)^r \mathrm{trace}(h_r(f,p))$$

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If p is not attracting $\Rightarrow L \neq \emptyset \Rightarrow H_0(N,L) = 0 \Rightarrow \mathrm{trace}(h_0(f^n,p)) = 0$.
If p is not repelling $\Rightarrow \partial N$ not totally contained in $L \Rightarrow H_d(N,L) = 0 \Rightarrow \mathrm{trace}(h_d(f^n,p)) = 0$.

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Except for attractors and repellers:

In dimension $\mathbf{d = 2}$, $\boxed{i(f^n,p) = -\mathrm{trace}(h_1(f^n,p))}$

In dimension $\mathbf{d = 3}$, $\boxed{i(f^n,p) = -\mathrm{trace}(h_1(f^n,p)) + \mathrm{trace}(h_2(f^n,p))}$

Description of the first Conley index

Theorem (Traces of $h_1(f^n, p)$)

There exists J finite and $\varphi : J \to J$ such that

$$\mathrm{trace}(h_1(f^n, p)) = -1 + \#\mathrm{Fix}(\varphi^n).$$

Idea

$$\begin{array}{c} H_1(N/L) \\ \Big\downarrow \bar{f}_{*,1} \\ H_1(N/L) \end{array}$$

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Idea

Long exact sequence of homology of the pair (N, L)

$$\begin{array}{ccccccccc}
\longrightarrow & H_1(N) & \longrightarrow & H_1(N, L) & \xrightarrow{\partial} & H_0(L) & \longrightarrow \\
 & \vdots & & \downarrow \bar{f}_{*,1} & & \vdots & \\
\longrightarrow & H_1(N) & \longrightarrow & H_1(N, L) & \xrightarrow{\partial} & H_0(L) & \longrightarrow
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Theorem

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a **continuous** *map and p a locally maximal fixed point, neither a sink nor a source. Then, $\exists J$ finite and $\varphi : J \to J$ such that*

$$i(f^n, p) = 1 - \#\mathrm{Fix}(\varphi^n).$$

In particular, always $i(f^n, p) \leq 1$ and $\{i(f^n, p)\}_{n \geq 1}$ is periodic.

Up to dimension 3 $\rightarrow$ homeomorphisms

$$i(f^n, p) = -\mathrm{trace}(h_1(f^n, p)) + \mathrm{trace}(h_2(f^n, p)).$$

Szymczak's duality

$$h_2(f, p) \cong \pm h_1(f^{-1}, p).$$

$\pm$ depends on whether f is orientation preserving or reversing.

φ describes $h_1(f, p)$ and φ' describes $h_1(f^{-1}, p)$

$$i(f^n, p) = -(-1 + \#\mathrm{Fix}(\varphi^n)) + (\pm 1)^n(-1 + \#\mathrm{Fix}(\varphi'^n)).$$

Theorem

φ dominates higher Conley indices: if $\mathrm{Fix}(\varphi) = \emptyset$ then $\mathrm{trace}(h_2(f, p)) = 0$

Results in dimension 3

Theorem

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a **homeomorphism** *and p a locally maximal fixed point, neither a sink nor a source. Then,*

- *(Le Calvez – Ruiz del Portal – Salazar) $\{i(f^n, p)\}_{n\geq 1}$ is periodic.*
- *If f is* **orientation–reversing** *then*

$$i(f, p) \leq 1.$$

Corollary

$\nexists$ *minimal orientation–reversing homeomorphisms in $\mathbb{R}^3$.*

Comments, questions

- We can **replace fixed point** p by an **invariant acyclic continua** X (only need triviality in the homological sense)

- In case X is **not acyclic**, i.e., $\check{H}(X) \neq 0$ then we should have in $\mathbb{R}^3$

$$i(f^n, X) = \text{periodic term } + \mathrm{trace}(\check{f}^n_{*,1} : \check{\mathrm{H}}_1(X) \to \check{\mathrm{H}}_1(X)).$$

- Is there any restriction to $\{i(f^n, p)\}_{n \geq 1}$ for continuous $f : \mathbb{R}^2 \to \mathbb{R}^2$ and p **isolated as a periodic point**?