Lexicographic infimax sequences

Philip Boyland, André de Carvalho, and Toby Hall

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Digit frequency sets of β-shifts

Fix $k \geq 2$, and let $\Sigma = \{1, 2, \dots, k\}^{\mathbb{N}}$, ordered lexicographically.

For $w \in \Sigma$, write $\mathrm{df}(w) \in \Delta$ for the frequency vector of the digits $1, \dots, k$, if it exists, where $\Delta \subset \mathbb{R}^k$ is the standard $(k-1)$-simplex.

$\mathcal{M} \subset \Sigma$ is the set of *maximal* sequences: $\sigma^r(w) \leq w$ for all $r \geq 0$.

Given $w \in \mathcal{M}$, the associated *symbolic β-shift* is $X(w) = \{v \in \Sigma : \sigma^r(v) \leq w \text{ for all } r \geq 0\}$.

Write $\mathrm{DF}(w) = \{\mathrm{df}(v) : v \in X(w) \text{ has a frequency vector}\} \subset \Delta$.

Question: What is $\mathrm{DF}(w)$?

Motivation

- As Phil Boyland explained in his talk, *calculation of rotation sets* in a specific family of torus homeomorphisms can be reduced to the calculation of $\mathrm{DF}(w)$ in the case $k = 3$. Many of the results in his talk are consequences of the results in this talk.
- Taking higher values of k gives analogous results for families of homeomorphisms of *higher-dimensional tori*.
- The results can also be applied to digit frequency sets of *greedy β-expansions* of real numbers $x \in [0, 1]$,

$$x = \sum_{r=1}^{\infty} \frac{w_r}{\beta^r},$$

where $\beta > 1$ and $0 \le w_r \le k - 1 = \lceil \beta \rceil - 1$.

Rewriting the problem

$X(w) = \{v \in \Sigma \,:\, \sigma^r(v) \le w$ for all $r \ge 0\}$.

$\mathrm{DF}(w) = \{\mathrm{df}(v) \,:\, v \in X(w)$ has a frequency vector$\} \subset \Delta$.

Given $\alpha \in \Delta$, let $\mathcal{R}(\alpha) = \{v \in \Sigma \,:\, \mathrm{df}(v) = \alpha\}$, and
$\mathcal{M}(\alpha) = \mathcal{M} \cap \mathcal{R}(\alpha)$ (maximal sequences with frequency α).

Define $\mathcal{I}(\alpha) = \inf \mathcal{M}(\alpha) \in \Sigma$, the *$\alpha$-infimax sequence*.

$$= \inf_{v \in \mathcal{R}(\alpha)} \sup_{r \ge 0} \sigma^r(v).$$

Then

$$\mathrm{DF}(w) = \{\alpha \in \Delta \,:\, \mathcal{I}(\alpha) \le w\},$$

so we aim to calculate $\mathcal{I}(\alpha)$ for each $\alpha \in \Delta$.

If $\mathcal{I}(\alpha) \in \mathcal{M}(\alpha)$ then we call it the *α-minimax sequence*.

The case $k = 2$

When $k = 2$ the simplex Δ is one-dimensional, and for each $\alpha = (1 - \alpha_2, \alpha_2) \in \Delta$ it is well known that

$$\mathcal{I}(\alpha) = s_{\alpha_2},$$

the *Sturmian sequence* of rotation α_2.

These are all *minimax* sequences, i.e. $\mathrm{df}(s_{\alpha_2}) = \alpha$.

For $k \geq 3$ the situation is more complicated. We focus on $k = 3$: most of what we do generalises naturally to higher values of k.

The finite problem

There is a finite version of the problem which is easily solved and which provides some insight into the general case.

We consider words W over the digits $1, 2, \ldots, k$. Such a word is *maximal* if $\overline{W} \in \Sigma$ is maximal, i.e. if W is at least as large as all of its cyclic permutations.

Let $\widehat{\Delta} = \{a = (a_1, \ldots, a_k) \in \mathbb{N}^k \, : \, a_k > 0\}$.

For each $a \in \widehat{\Delta}$, write

$$\begin{aligned} \widehat{\mathcal{R}}(a) &= \{\text{Words } W \, : \, W \text{ has } a_i \; i^{\text{s}} \text{ for each } i\} \qquad \text{(a finite set)}, \\ \widehat{\mathcal{M}}(a) &= \{W \in \widehat{\mathcal{R}}(a) \, : \, W \text{ is maximal}\}, \qquad \text{and} \\ \widehat{\mathcal{I}}(a) &= \min \widehat{\mathcal{M}}(a). \end{aligned}$$

How do we calculate $\widehat{\mathcal{I}}(a)$? e.g. $\widehat{\mathcal{I}}(24, 3, 14) = 31(311)^{10}(312)^3$ is the smallest maximal word with 24 1^{s}, 3 2^{s}, and 14 3^{s}.

Solution of the finite problem (Case $k = 3$)

Let W be the smallest maximal word with a_1 1s, a_2 2s, and a_3 3s.

Let $n = \lfloor a_1/a_3 \rfloor \geq 0$, so that $n\, a_3 \leq a_1 < (n+1)\, a_3$.

$W = 31^n W_1\, 31^n W_2 \cdots 31^n W_{a_3}$, where W_r are words in 1 and 2.

Each W_r is of the form $1^{p_r} 2^{q_r}$.

Some $p_r = 0$, and in particular $p_1 = 0$ since W is maximal.

Every $p_r \leq 1$. For suppose that $p_s \geq 2$ for some least $s \geq 2$.

Push one of the 1s at the start of W_s to the start of W_{s-1}. Then every cyclic permutation starting with the letter 3 becomes smaller, with the exception of the one before W_s, which can't be maximal. (e.g. $31\,312222\,3111 \;\rightarrow\; 31\,3112222\,311$.)

$W = \widehat{\mathcal{I}}(a)$ is a concatenation of the words 31^n, 31^{n+1} and 2.
(where $n = \lfloor a_1/a_3 \rfloor$)

Solution of the finite problem (continued)

For $n \geq 0$, let Λ_n be the substitution

$$\Lambda_n : \begin{cases} 1 & \mapsto \quad 2, \\ 2 & \mapsto \quad 3\,1^{n+1}, \\ 3 & \mapsto \quad 3\,1^n. \end{cases}$$

We've showed that $\widehat{\mathcal{I}}(a) = \Lambda_{\lfloor a_1/a_3 \rfloor}(V)$ for some word V.

By linear algebra, the number of each letter in V is given by

$$\widehat{K}_n(a) = (a_2,\;\; a_1 - na_3,\;\; (n+1)a_3 - a_1).$$

It can easily be shown that each Λ_n is order-preserving, and that the set of words whose image under Λ_n lies in $\widehat{\mathcal{M}}(a)$ is exactly $\widehat{\mathcal{M}}(\widehat{K}_n(a))$, so that

$$\widehat{\mathcal{I}}(a) = \Lambda_n(\widehat{\mathcal{I}}(\widehat{K}_n(a))), \qquad \text{where } n = \lfloor a_1/a_3 \rfloor.$$

Solution of the finite problem (example)

- If $a_1 = a_2 = 0$, then $\widehat{\mathcal{I}}(a) = 3^{a_3}$.
- Otherwise, $\widehat{\mathcal{I}}(a) = \Lambda_n(\widehat{\mathcal{I}}(\widehat{K}_n(a)))$, where $n = \lfloor a_1/a_3 \rfloor$.

(Recall $\widehat{K}_n(a) = (a_2,\ \ a_1 - na_3,\ \ (n+1)a_3 - a_1)$

and $\Lambda_n \colon 1 \mapsto 2,\ \ 2 \mapsto 31^{n+1},\ \ 3 \mapsto 31^n$.)

Example $\qquad a = (24, 3, 14)$.

$$(24,3,14) \xrightarrow{\widehat{K}_1} (3,10,4) \xrightarrow{\widehat{K}_0} (10,3,1) \xrightarrow{\widehat{K}_{10}} (3,0,1) \xrightarrow{\widehat{K}_3} (0,0,1).$$

$$3 \xrightarrow{\Lambda_3} 31^3 \xrightarrow{\Lambda_{10}} 31^{10}2^3 \xrightarrow{\Lambda_0} 32^{10}(31)^3 \xrightarrow{\Lambda_1} 31(311)^{10}(312)^3.$$

So $\mathcal{I}(24,3,14) = 31(311)^{10}(312)^3$.

Review

$\widehat{\Delta} = \{a \in \mathbb{N}^3 \,:\, a_3 > 0\}$ is partitioned into subsets
$\widehat{\Delta}_n = \{a \in \Delta \,:\, \lfloor a_1/a_3 \rfloor = n\}$ for $n \geq 0$.

Linear bijection $\widehat{K}_n \colon \widehat{\Delta}_n \to \widehat{\Delta}$ given by
$\widehat{K}_n(a) = (a_2,\ \ a_1 - na_3,\ \ (n+1)a_3 - a_1)$.

Alternatively, ∞ to 1 map $\widehat{K} \colon \widehat{\Delta} \to \widehat{\Delta}$ given by $\widehat{K}(a) = \widehat{K}_{\widehat{J}(a)}(a)$, where $\widehat{J}(a) = \lfloor a_1/a_3 \rfloor$.

Itinerary map $\widehat{\Phi} \colon \widehat{\Delta} \to \mathbb{N}^{\mathbb{N}}$ given by $\widehat{\Phi}(a)_r = \widehat{J}(\widehat{K}^r(a))$.

$\widehat{\Phi}(a) = n_0 n_1 \dots n_r 0^\infty$, since iteration always ends at the fixed point $(0,0,*) \in \widehat{K}_0$.

$\widehat{\mathcal{I}}(a) = \Lambda_{n_0}\Lambda_{n_1}\cdots\Lambda_{n_r}(3^*) = \Lambda_{n_0}\Lambda_{n_1}\cdots\Lambda_{n_r}\Lambda_0^\infty(3^*)$.

By analogy

$\Delta = \{\alpha \in \mathbb{R}^3_{\geq 0} : \alpha_3 > 0, \sum \alpha_i = 1\}$ is partitioned into subsets $\Delta_n = \{\alpha \in \Delta : \lfloor \alpha_1/\alpha_3 \rfloor = n\}$ for $n \geq 0$.

Projective homeomorphism $K_n \colon \Delta_n \to \Delta$ given by

$$K_n(\alpha) = \left(\frac{\alpha_2}{1-\alpha_1}, \ \frac{\alpha_1 - n\alpha_3}{1-\alpha_1}, \ \frac{(n+1)\alpha_3 - \alpha_1}{1-\alpha_1} \right).$$

Alternatively, ∞ to 1 map $K \colon \Delta \to \Delta$ given by $K(\alpha) = K_{J(\alpha)}(\alpha)$, where $J(\alpha) = \lfloor \alpha_1/\alpha_3 \rfloor$.

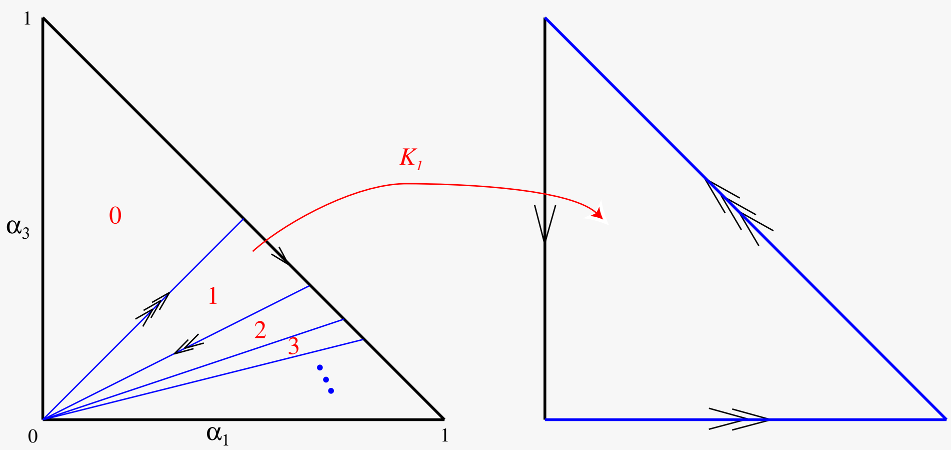
Itinerary map $\Phi \colon \Delta \to \mathbb{N}^{\mathbb{N}}$ given by $\Phi(\alpha)_r = J(K^r(\alpha))$.

$S \colon \mathbb{N}^{\mathbb{N}} \to \Sigma$ given by $S(\mathbf{n}) = \lim_{r\to\infty} \Lambda_{n_0}\Lambda_{n_1}\cdots\Lambda_{n_r}(3^\infty)$.

$\mathcal{I}(\alpha) = S \circ \Phi(\alpha)$ by analogy with finite case: a true statement, though the proof is less straightforward. When $\alpha \in \mathbb{Q}^3$, reduces to finite case. ($\alpha \in \mathbb{Q}^3$ if and only if $\Phi(\alpha) = n_0 n_1 \ldots n_r \overline{0}$.)

Each $\mathcal{I}(\alpha)$ is *almost periodic*, so the orbit closure is *minimal*.

Multidimensional continued fraction algorithm



$$K_n(\alpha) = \left(\frac{\alpha_2}{1-\alpha_1}, \ \frac{\alpha_1 - n\alpha_3}{1-\alpha_1}, \ \frac{(n+1)\alpha_3 - \alpha_1}{1-\alpha_1} \right).$$

Exceptional proportions

- The itinerary map $\Phi\colon \Delta \to \mathbb{N}^{\mathbb{N}}$ is *not injective*.
- There are intervals in Δ all of which have the same itinerary — and hence the *same infimax*.
- In these intervals, therefore, the infimax is *not a minimax*.
- We say that α is *exceptional* if it shares its itinerary with other points of Δ, and *regular* otherwise.

Theorem Let $\alpha \in \Delta$ have itinerary $\mathbf{n}$.

- $\mathcal{I}(\alpha)$ is a minimax if and only if α is regular.
- If $0 < n_r < Cr^2$ for all r then α is regular.
- If $n_r \geq 2^{r+2}\prod_{i=0}^{r-1}(n_i+2)$ for all $r \geq 1$ then α is exceptional. For example if $n_r = 2^{2^{3r}}$.

The bound in the final part can easily be improved, but describing the boundary between the regular and exceptional cases is probably difficult (experimentally, $n_r = r^3$ is exceptional).

Bruin and Troubetzkoy prove that, when $k=3$, $\mathbf{n}$ is exceptional if $n_{r+1} \geq Cn_r$ for some $C>1$. (Not enough to have $n_r \geq C^r$.)

Back to the original problem

For $w \in \mathcal{M}$, we have $X(w) = \{v \in \Sigma \,:\, \sigma^r(v) \leq w \text{ for all } r \geq 0\}$,

$$\begin{aligned}\mathrm{DF}(w) &= \{\mathrm{df}(v) \,:\, v \in X(w) \text{ has a frequency vector}\}\\ &= \{\alpha \in \Delta \,:\, \mathcal{I}(\alpha) \leq w\}.\end{aligned}$$

As w increases, $\mathrm{DF}(w)$ changes whenever w passes through an element of the set $\mathcal{IM}$ of infimax sequences.

The map $S\colon \mathbb{N}^{\mathbb{N}} \to \mathcal{IM}$ is an *order-preserving homeomorphism* when $\mathbb{N}^{\mathbb{N}}$ is ordered reverse lexicographically.

$\mathbb{N}^{\mathbb{N}}$ is a *Cantor set* $\mathcal{N}$ less the right hand endpoints of gaps, which have left-hand endpoints of the form $n_0 \dots n_r 0^\infty$ (rational case).

We therefore get *locally constant* digit frequency sets as w moves through one of these gaps: they are polygons, with $N+3$ vertices, where N is the number of non-zero n_i $(1 \leq i \leq r)$.

Generic itineraries are regular and correspond to *totally irrational* α.

Explicit description of digit frequency sets I

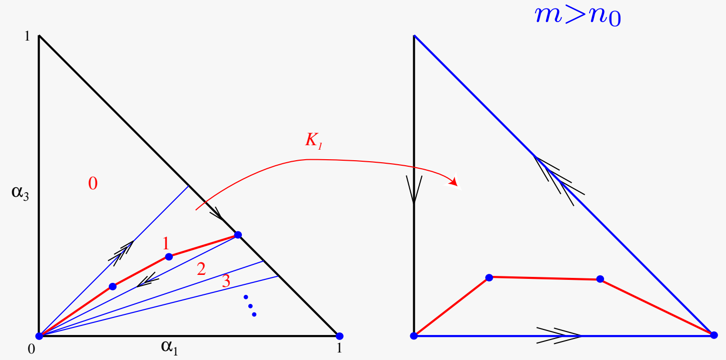
For $w \in \mathcal{M}$, we have $X(w) = \{v \in \Sigma : \sigma^r(v) \leq w \text{ for all } r \geq 0\}$,

$$\begin{aligned} \mathrm{DF}(w) &= \{\mathrm{df}(v) : v \in X(w) \text{ has a frequency vector} \\ &= \{\alpha \in \Delta : \mathcal{I}(\alpha) \leq w\}. \end{aligned}$$

Writing $\mathbf{n} = \mathbf{n}(w) = \max\{\mathbf{m} \in \mathcal{N} : S(\mathbf{m}) \leq w\}$ we have $\mathrm{DF}(w) = \mathrm{DF}(S(\mathbf{n})) = \{\alpha \in \Delta : \Phi(\alpha) \leq \mathbf{n}\} =: \mathrm{DF}(\mathbf{n})$.

If $\Phi(\alpha)_0 = n_0$ then $\Phi(\alpha) \leq \mathbf{n} \iff \Phi(K_{n_0}(\alpha)) \leq \sigma(\mathbf{n})$, so we get

$$\mathrm{DF}(\mathbf{n}) = K_{n_0}^{-1}(\mathrm{DF}(\sigma(\mathbf{n}))) \cup \bigcup_{m>n_0} \Delta_m.$$



Explicit description of digit frequency sets II

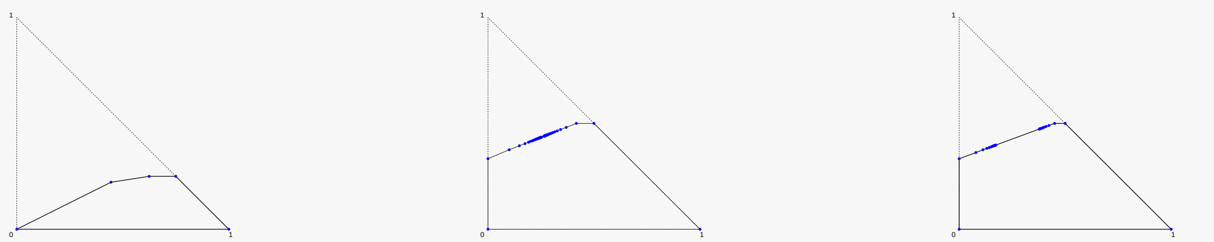
The extreme points of $\mathrm{DF}(\mathbf{n})$ are obtained from those of $\mathrm{DF}(\sigma(\mathbf{n}))$, together with one extra point unless $n_1 = 0$.

The extreme points of $\mathrm{DF}(\mathbf{n})$ are $(0,0,1)$, $(0,1,0)$, together with $K_{n_0}^{-1} \circ K_{n_1}^{-1} \circ \cdots \circ K_{n_r}^{-1}(0,1,0)$, for each r with $n_{r+1} \neq 0$, and either one (regular) or two (exceptional) additional extreme points.

$\mathrm{DF}(\mathbf{n})$ is a *polygon* $\iff$ $\mathbf{n} = n_0 n_1 \dots n_r \overline{0}$

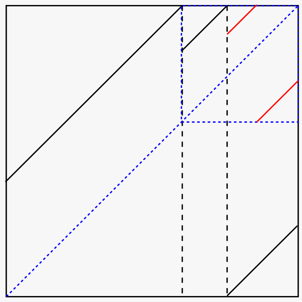
$\iff$ the infimax $S(\mathbf{n})$ has rational digit frequency

$\iff$ $\mathbf{n}$ is the left hand endpoint of a gap.



Infimax sequences and interval translation mappings

Bruin and Troubetzkoy (2003) consider a family of *interval translation mappings* T on 3 intervals.



- Renormalize on $I_2 \cup I_3$. I_2 returns immediately $(1 \mapsto 2)$.
- The right hand end of I_3 returns after n iterates $(3 \mapsto 31^n)$.
- The left hand end returns after $n+1$ iterates $(2 \mapsto 31^{n+1})$.
- The interesting case (when $\bigcap_{n \geq 0} T^n(I)$ is a Cantor set) is when T renormalizes infinitely often: in this case the itinerary of the right hand endpoint of I is a (non-rational) infimax sequence.