An Introduction to Out(F_n) Part I
(Relative) Train Track Maps

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Disclaimer

Emphasize aspects that are most accessible to a dynamics audience.

F_n

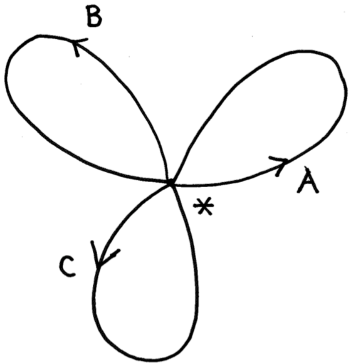
Finite alphabet $\{A, B, \ldots\}$

$\bar{A} = A^{-1}, \;\; \bar{B} = B^{-1}, \;\; \ldots$

$F_n =$ reduced finite words in $\{A, \bar{A}, B, \bar{B}, \ldots\}$

Identify F_n with $\pi_1(R_n, *)$.

Each element is represented by a closed path based at $*$.



Automorphisms $\Phi : F_n \to F_n$ are represented by homotopy equivalences of R_n that fix the basepoint.

Example: $\Phi : F_2 \to F_2$

$$A \mapsto ABA \qquad B \mapsto ABABA$$

represented by $f : R_2 \to R_2$ with the same description.

$\mathrm{Aut}(F_n) \leftrightarrow$

homotopy equivalence of R_n preserving $*$/ homotopy rel $*$

$\mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/\,\mathrm{Inn}(F_n)$

$\mathrm{Out}(F_n)$ - forget $*$

$\mathrm{Out}(F_n) \leftrightarrow$ homotopy equivalence of R_n / homotopy

This is too restrictive.

Example: $\Phi : F_2 \to F_2$

$$A \mapsto \bar{B} \qquad B \mapsto A\bar{B}$$

has period 3

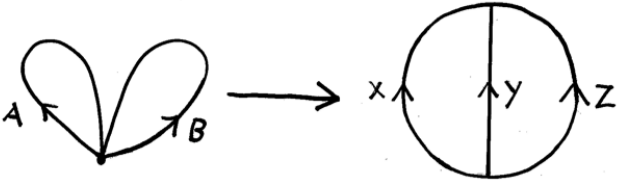
$$A \mapsto \bar{B} \mapsto B\bar{A} \mapsto A\bar{B}B = A$$

$$B \mapsto A\bar{B} \mapsto \bar{B}B\bar{A} = \bar{A} \mapsto B$$

but $f : R_2 \to R_2$ is not periodic.

A marked graph is a core graph equipped with a homotopy equivalence $\rho : R_n \to G$

$$\rho \qquad A \mapsto X\bar{Y} \qquad B \mapsto Z\bar{Y}$$



The marking ρ identifies $\pi_1(G)$ with $\pi_1(R) = F_n$ (up to inner automorphism) and so

$\mathrm{Out}(F_n) \leftrightarrow$ homotopy equivalence of G / homotopy

Continuing Example: $\Phi: \quad A \mapsto \bar{B} \quad B \mapsto A\bar{B}$

$f: G \to G$ defined by

$$X \mapsto Y \mapsto Z \mapsto X$$

represents ϕ.

Automorphisms act on elements and subgroups of F_n.

Outer automorphisms act on conjugacy classes of elements and subgroups of F_n.

$f : G \to G$ represents $\phi \in \mathrm{Out}(F_n)$ if it induces the same action on conjugacy classes of elements.

Other spaces with $\pi_1(X) = F_n$.

Example: A surface S with connected non-empty ∂S

$\mathrm{MCG}(S) =$ Homeo(S)/ isotopy = Homeo(S)/ homotopy

$$Homeo(S) \to \mathrm{Out}(\pi_1(S)) = \mathrm{Out}(F_n)$$

induces

$$\mathrm{MCG}(S) \hookrightarrow \mathrm{Out}(F_n)$$

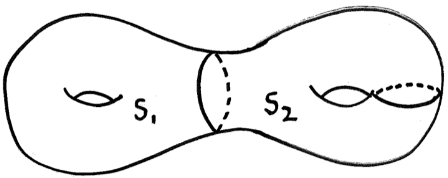
Thurston classification theorem

Assume $\chi(S) < 0$ and not a pair of pants.

Definition 1

$\mu \in \mathrm{MCG}(S)$ is reducible if it preserves a non-empty multicurve $\mathcal{C}$.

In that case each $\mu|S_i$ is a well defined element of $\mathrm{MCG}(S_i)$



Theorem 1 (Thurston)

If μ is irreducible then μ is represented by a pseudo-Anosov homemorphism.

The reducible case is handled by recursion.

There is a canonical μ-invariant multicurve $\mathcal{S}$ with the minimal number of complementary components such that each $\mu|S_i$ is either trivial or irreducible.

What is the analogue for Out(F_n)?

What properties of $f : G \to G$ do we want?

We are often interested in the action of ϕ on conjugacy classes $[a]$ of F_n.

$$\text{conjugacy classes } [a] \leftrightarrow \text{circuits } \sigma \subset G$$

$f(\sigma)$ can be tightened to a circuit $f_\#(\sigma)$ and similarly for a path with endpoints at vertices.

We want to minimize tightening.

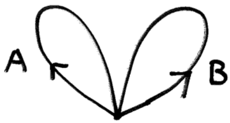
$f : G \to G$ is a train track map if f^k restricts to an immersion on each edge E of G for all $k \geq 1$.

In other words, no tightening is necessary when iterating edges.

Example 2

Positive Automorphism

$$A \mapsto ABA \qquad B \mapsto ABABA$$



Directions at $*$

$$A \qquad B \qquad \bar{A} \qquad \bar{B}$$

A turn is an unordered pair of directions with a common basepoint.

The edge path $AA\bar{B}A\ldots$ takes the turns $(\bar{A}, A), (\bar{A}, \bar{B}), (B, \bar{A})\ldots$

Nondegenerate turns

$$(A, B) \qquad (A, \bar{A}) \qquad (A, \bar{B}) \qquad (B, \bar{A}) \qquad (B, \bar{B}) \qquad (\bar{A}, \bar{B})$$

Degenerate turns

$$(A, A) \qquad (B, B) \qquad (\bar{A}, \bar{A}) \qquad (\bar{B}, \bar{B})$$

An edge path is immersed if and only if it that makes only non-degenerate turns.

Assuming that f is an immersion on each edge there is an induced map Df on directions and turns

$$A \mapsto A \qquad B \mapsto A \qquad \bar{A} \mapsto \bar{A} \qquad \bar{B} \mapsto \bar{A}$$

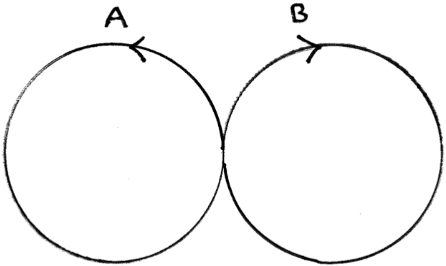
A turn is illegal if it is mapped by some iterate to a degenerate path.

Illegal turns

$$(A, B) \mapsto (A, A) \qquad (\bar{A}, \bar{B}) \mapsto (\bar{A}, \bar{A})$$

Note: Legal turns are mapped to legal turns.

A path is legal if it takes only legal turns.



The following are equivalent.

1. $f(E)$ is a legal path for each edge E.
2. f maps legal paths to legal paths.
3. $f^k|E$ is an immersion for each $k \geq 1$ and each edge E.

To each $f : G \to G$ there is a transition matrix $M(f)$.

Example 3

$$A \mapsto ABA \qquad B \mapsto ABABA$$

$$M(f) = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

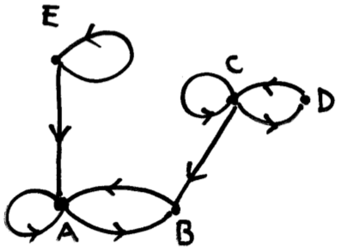
If $f : G \to G$ is a train track map then $(M(f))^k = M(f^k)$

There is also a transition graph $\Gamma(f)$

Example 4

$A \mapsto BA \quad B \mapsto A \quad C \mapsto DBC \quad D \mapsto C \quad E \mapsto EA$

$G_1 = (A, B) \quad G_2 = (A, B, C, D) \quad G_3 = G$



I am going to assume that each vertex in $\Gamma(f)$ is contained in at least one cycle.

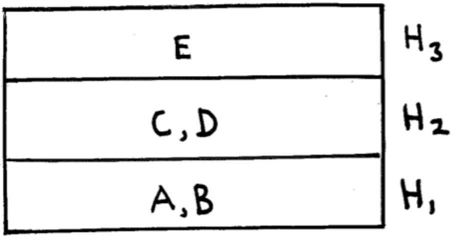
$\Gamma(f)$ determines a filtration

$$\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G$$

by invariant subgraphs built up by starting with an invariant subgraph and then adding strata

$$H_i = G_i - G_{i-1}$$

Strata correspond to equivalence classes of vertices in $\Gamma(f)$.



There are two types of strata for rotationless ϕ

An NEG stratum H_i has one edge E_i and

$$f(E_i) = u_i E_i v_i$$

for paths $u_i, v_i \subset G_{i-1}$

An EG stratum H_r has multiple edges and its transition submatrix has a positive iterate (and so a PF eigenvalue > 1).