Individual Elements of Out(F_n)

1. What are the possible growth rates for the action of ϕ on conjugacy classes $[a]$?

2. Suppose that Φ is an automorphism. Is the fixed subgroup $\mathrm{Fix}(\Phi) = \{a \in F_n : \Phi(a) = a\}$ finitely generated? What can its rank be?

3. How can one tell if ϕ is geometric? (Realized by a pseudo-Anosov homeomorphisms of a surface with one boundary component?)

4. What is the correct notion of irreducible?

5. What properites should an $f : G \to G$ representing an irreducible ϕ have?

6. What about the reducible case? Does it follow from the irreducible case?

Reducibility

A subgroup A of F_n is a free factor if there exists a subgroup B such that $F_n = A * B$.

Equivalently, A is realized by a subgraph of a marked graph.

$\phi \in \mathrm{Out}(F_n)$ is reducible if it preserves (the conjugacy class $[A]$ of) a free factor A

Equivalently, ϕ is represented by $f : G \to G$ in which f preserves a proper subgraph.

In that case each $\phi|[A]$ is a well defined element of $\mathrm{Out}(A)$

Bad news: There need not be an invariant complementary free factor.

Theorem 1 (BH)

Each irreducible $\phi \in \mathrm{Out}(F_n)$ is represented by an (irreducible) train track map.

Proof (Original) : Minimize the entropy.

If $f : G \to G$ is not a train track map then there is a procedure to find a new $f : G \to G$ with smaller PF eigenvalues. This stops after a finite number of iterations.

Proof (Updated [B]) : Minimize the Lipschitz constant for $f : G \to G$.

Iteration of Conjugacy Classes

Motivate Train Track Property

Suppose that $f : G \rightarrow G$ is a train track map representing ϕ and that σ a circuit corresponding to $[a]$.

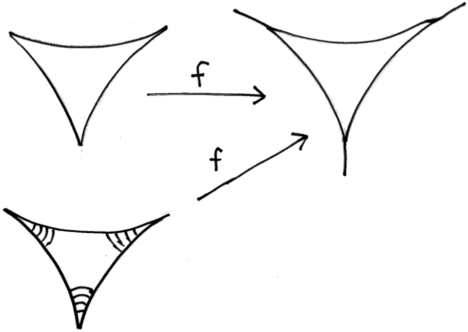
If σ is legal then $[a]$ grows exponentially with rate λ

Otherwise

$$\sigma = \sigma_1 \sigma_2 \ldots \sigma_p$$

where σ_i is legal and the indicated turns are illegal.

Can assume that the number of illegal turns in $f_\#^k(\sigma)$ is independent of k.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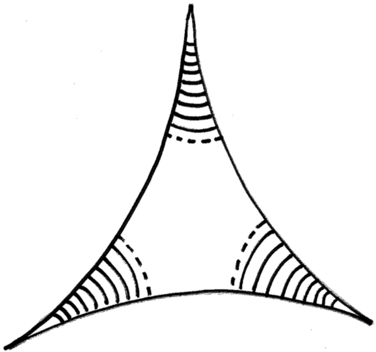
The lengths of the subpaths in $f(\sigma_i)$ that are tightened away is uniformly (independent of σ) bounded.

Iterate to form $f^k_\#(\sigma)$

The lengths of the subpaths in $\bar{\sigma}_i$ and σ_i that are identified is uniformly (independent of σ and k) bounded.



$\mathcal{P} = \{\rho$: each $f_\#^k(\rho)$ has exactly one illegal turn and uniformly bounded length $\}$

$\mathcal{P}$ is a finite $f_\#$-invariant set

Lemma 2

For every σ there exists K such that $f_\#^k(\sigma)$ has a splitting into legal subpaths and periodic elements of $\mathcal{P}$ for all $k \geq K$.

Corollary 3

Each $[a]$ is either ϕ-periodic or grows exponentially (with growth rate λ).

Proposition 1

Each irreducible ϕ is represented by an irreducible train track map $f : G \to G$ such that $\mathcal{P}$ has at most one periodic element ρ.

If there is such a ρ and if it closed then it crosses every edge of G exactly twice.

Corollary 4

If Φ represents ϕ then $\mathrm{Fix}(\Phi)$ has rank at most one.

Corollary 5

An irreducible ϕ is geometric if and only if it preserves a (necessarily unique) conjugacy class

Some Theorems

Theorem 6 (BH)

(Scott Conjecture) *The rank of* $\mathrm{Fix}(\Phi))$ *is* $\leq n$ *for all* $\Phi \in \mathrm{Aut}(F_n)$.

Example 7

$\Phi: \quad A \mapsto A \quad B \mapsto BA \quad C \mapsto CA^2$

$\mathrm{Fix}(\Phi) = \langle A, BA\bar{B}, CA\bar{C} \rangle$

Theorem 8 (BH)

For each $\phi \in \mathrm{Out}(F_n)$ and conjugacy class $[a]$ the length of $\phi^k([a])$ either grows polynomially of degree $\leq n-1$ or exponentially.

Subgroups of Out(F_n)

1. Does Out(F_n) satisfy the Tits Alternative? (Every finitely generated subgroup is either virtually abelian or contains a free group of rank ≥ 2.)?

2. For which $\phi, \psi \in$ Out(F_n) does there exist N such that $\langle \phi^N, \psi^N \rangle$ is free? Can one choose N independently of ϕ, ψ?

3. What do abelian subgroups look like?

4. Is it true that every finitely generated subgroup of Out(F_n) is either virtually abelian or has infinitely generated H_b^2?

5. Does every finitely generated irreducible subgroup contain an irreducible element?

Definition 9

A subgroup $\mathcal{H}$ of $\mathrm{Out}(F_n)$ is *irreducible* if there is no free factor whose conjugacy class is $\mathcal{H}$-invariant.

Theorem 10 (HM)

[Absolute version] If $\mathcal{H} < \mathrm{Out}(F_n))$ is finitely generated and irreducible then $\mathcal{H}$ contains an irreducible element.