An Introduction to Out(F_n) Part II
The Subgroup Decomposition Theorem

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Sao Paolo, April 2014

Subgroup Decomposition Theorem (Absolute Version)

Definition 1

A subgroup $\mathcal{H}$ of $\mathrm{Out}(F_n)$ is *irreducible* if there is no free factor whose conjugacy class is $\mathcal{H}$-invariant.

Theorem 2 (HM)

[Absolute version] If $\mathcal{H} < \mathrm{Out}(F_n)$) is finitely generated and irreducible then $\mathcal{H}$ contains an irreducible element.

Simplifying Assumptions:

- ϕ acts trivially on $\mathbb{Z}/3\mathbb{Z}$ homology
- $f : G \to G$ is a train track map.
- All subgraphs are connected. (Algebraically, all free factor systems are free factors)
- All EG strata are non-geometric.

Recall that $f : G \to G$ has a transition graph $\Gamma(f)$ and a filtration by invariant subgraphs.

Attracting Laminations

A line γ in a marked graph G is a bi-infinite immersed edge path.

A lamination is a closed set of lines.

Theorem 3 (BFH)

Each EG stratum determines an attracting lamination Λ^+ and a repelling lamination Λ^-.

Roughly speaking, Λ^+ is the closure of the unstable manifold of a periodic point.

Example 4

$A \mapsto ABCA \qquad B \mapsto BCA \qquad C \mapsto CBCBC$

Iterate C and focus on middle copy of C (or fixed point in middle of C

$C \mapsto CBCBC \mapsto (CBCBC)(BCA)(CBCBC)(BCA)(CBCBC) \mapsto \dots$

We have an increasing sequence of paths whose limit is an invariant line. The closure of this line is Λ^+.

This is independent of which periodic point you start with.

Weak Topology on Lines

Neighborhood basis for a line $\gamma \subset G$:

Choose an exhaustion $\gamma_0 \subset \gamma_1 \subset \gamma_2 \subset \ldots$ of γ.

$\sigma \in N(\gamma_k)$ if γ_k is an unoriented subpath of σ.

This can be made independent of G by using the marking and universal covers.

Lemma 5 (Cooper)

Suppose that $f : G_1 \to G_2$ is a homotopy equivalence and that $\tilde{f} : \tilde{G}_1 \to \tilde{G}_2$ is a lift. Then there is a constant $C(f)$ such that:

For any finite path $\tilde{\sigma}$, with endpoints say $\tilde{x}$ and $\tilde{y}$, the image $\tilde{f}(\tilde{\sigma})$ is contained in the $C(f)$-neighborhood of the path $\tilde{f}_\#(\tilde{\sigma})$ connecting $\tilde{f}(\tilde{x})$ to $\tilde{f}(\tilde{y})$.

Corollary 6

Same hypotheses. For any line $\tilde{L} \subset \tilde{G}_1$ there is a unique line $\tilde{f}_\#(\tilde{L}) \subset \tilde{G}_2$ such that $\tilde{f}(\tilde{L}_1)$ is contained in the $C(f)$-neighborhood of $\tilde{f}_\#(\tilde{L}_1)$.

Corollary 7

Same hypotheses. Let $\tau := f_{\#\#}(\sigma) \subset G_2$ be the path obtained from $f_\#(\sigma)$ by removing the initial and terminal subpaths of length $C(f)$. Then $f_\#(N(\sigma)) \subset N(\tau)$.

Corollary 8

Λ^+ has an attracting neighborhood.

What is the basin of attraction? Better, what is its complement of the basin?

Definition 9

A path of the form $f^k(E)$ is a k-tile or just a tile if k is unspecified.

Lemma 10

A line is a leaf of Λ^+ if and only if each of its subpaths is contained in a tile.

Lemma 11

A circuit σ is weakly attracted to Λ^+ if and only if for each k there exists M such that $f^m_\#(\sigma)$ contains a k-tile for all $m \geq M$.

Lemma 12

If ϕ is irreducible and non-geometric then the action on lines has N-S dynamics.

$\Lambda_r^+ \longleftrightarrow H_r$

$Z_r =$ subgraph of G whose edges E satisfy: there is no oriented path in $\Gamma(f)$ from the vertex representing E to a vertex representing an edge in H_r.

A circuit is NOT attracted to Λ_r^+ if and only if it is contained in Z_r.

$NA(\Lambda)$ is the free factor corresponding to Z_r

There is another useful invariant of Λ_r^+.

The free factor support $FFS(\Lambda_r^+)$ is the smallest free factor that contains Λ_r^+.

One can arrange that $FFS(\Lambda_r^+) = G_r$ for any one EG stratum.

Lemma 13

ϕ is irreducible if and only if it has an attracting lamination Λ^+ such that $NA(\Lambda^+)$ is trivial and $FFS(\Lambda^+) = [F_n]$.

Example 14

$f : R_5 \to R_5$ fixes A, B, C and $D \mapsto DwE$ and $E \mapsto DDE$.

$\Lambda^+ \longleftrightarrow (D, E)$ stratum

If w is complicated then $FFS = [F_n]$ and $NA = [\langle A, B, C\rangle]$.

Example 15

$f : R_4 \to R_4$ $\qquad A, B, C$ is an EG stratum and $D \mapsto Dw$.

$\Lambda^+ \longleftrightarrow (A, B, C)$ stratum

If w is non-trivial then $FFS = [\langle A, B, C\rangle]$ and NA is trivial.

Strategy of Proof

Find EG Prove that $\mathcal{H}$ contains at least one element with exponential growth.

Reduce NA Prove that if $\phi \in \mathcal{H}$ has an attracting lamination Λ^+_ϕ and $NA(\Lambda_\phi)$ has rank $R > 0$ then there exists $\xi \in \mathcal{H}$ and Λ^+_ξ such that $NA(\Lambda_\xi)$ has rank $< R$.

Make FFS Bigger Prove that if $\phi \in \mathcal{H}$ and $NA(\Lambda_\phi)$ is trivial and $FFS(\Lambda^+_\phi)$ has rank has rank $S < n$ then there exists $\xi \in \mathcal{H}$ and Λ^+_ξ such that $NA(\Lambda_\xi)$ is trivial and $FFS(\Lambda^+_\xi)$ has rank has rank $> S$

In the $\mathrm{MCG}(S)$ this amounts to showing:

If D_α and D_β are Dehn twists and if α crosses β then $D_\alpha^m D_\beta^n$ has a pseudo-Anosov component for some (all) large m, n.

If S_ϕ is a pseudo-Anosov component for ϕ and S_ψ is a pseudo-Anosov component for ψ and if S_1 crosses S_2 then $S_1 \cup S_2$ is contained in a pseudo-Anosov component for $\phi^m \psi^n$ for some (all) large m, n.

To find an EG element we apply

Theorem 16 (Kolchin Theorem)

[BFH] If every element of $\mathcal{H}$ is UPG then there is an $\mathcal{H}$-invariant filtration $\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G$ where each stratum has one edge.

Reducing NA

Start with ϕ and Λ_ϕ^+ with $NA(\phi)$ a proper free factor.

Example 17

$$A \mapsto AB \qquad B \mapsto BAB \qquad C \mapsto CD \qquad D \mapsto DACD$$

$$\Lambda^+ \longleftrightarrow (C, D) \qquad NA = (A, B)$$

Using irreducibility of $\mathcal{H}$ choose $\theta \in \mathcal{H}$ such that $\theta(NA(\phi)) \neq NA(\phi)$.

Let $\psi = \theta\phi\theta^{-1}$ so $\Lambda^+_\psi = \theta(\Lambda^+_\phi)$ and $NA(\psi) = \theta(NA(\phi))$.

We are interested in $\xi = \phi^k \psi^l$ for $k, l > K$ for some large K.

Draw abstract picture.

Lemma 18

$\theta(\Lambda_r^+)$ is weakly attracted to Λ_r^+.

Proof.

Can assume that $FFS(\Lambda_\phi^+)$ is realized by a subgraph G_r.

The homology assumption implies that $\theta(G_r)$ crosses every edge in G_r.

This implies that $\theta(\Lambda_r^+)$ crosses every edge in G_r and hence is not contained in $NA(\Lambda_r^+)$. □

Lemma 19

There is an attracting neighborhood U^+_ϕ for Λ^+_ϕ with the following property:

For any neighborhood V_ϕ of Λ_ϕ we have $\xi(U_\phi) \subset V_\phi$ for all sufficiently large K.

Write $U_\phi = N(\alpha_0)$ for some subpath α_0 of Λ^+_ϕ.

Sinc Λ^+_ϕ is birecurrent we can choose a subpath α_1 that contains three disjoint copies of α_0.

Choose $V_\phi = N(\alpha_1)$.

Interpretation: $f_{\#\#}(\alpha_0)$ contains three disjoint copies of α_0.

Construct attracting invariant line and Λ^+_ξ by iteration as in the Example.

Lemma 20

$NA(\Lambda^+_\xi) \subset NA(\Lambda^+_\phi) \cap NA(\Lambda^+_\psi)$

Since $NA(\Lambda^+_\phi) \cap NA(\Lambda^+_\psi) = NA(\Lambda^+_\phi) \cap \theta(NA(\Lambda^+_\phi))$ this completes the proof because $NA(\Lambda^+_\phi)$ is not $\mathcal{H}$-invariant.