Conjugacy classes of pseudo-rotations and distortion elements in groups of homeomorphisms of surfaces

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1. Pseudo-rotations

2. Distortion elements in groups of homeomorphisms of surfaces

Lemma

Let $f \in \mathrm{Homeo}_0(\mathbb{S}^1)$ *with rotation number* α. *Then* f *has conjugates arbitrarily close to the rotation* R_α.

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Definition

A homeomorphism f in $\mathrm{Homeo}_0(\mathbb{A})$ is called a *pseudo-rotation* if there exists α in $\mathbb{R}$ and a lift $\tilde{f} : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ of f such that, for any point x of $\widetilde{\mathbb{A}}$,

$$\lim_{n \to +\infty} \frac{p_1(\tilde{f}^n(x))}{n} = \alpha,$$

where $p_1 : \tilde{\mathbb{A}} = \mathbb{R} \times [0,1] \to \mathbb{R}$ is the projection.

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The class of α in $\mathbb{R}/\mathbb{Z}$ is the *angle* of the pseudo-rotation f.

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A homeomorphism f in $\mathrm{Homeo}_0(\mathbb{T}^2)$ is called a *pseudo-rotation* if there exists α in $\mathbb{R}^2$ and a lift $\tilde{f} : \widetilde{\mathbb{T}^2} \to \widetilde{\mathbb{T}^2}$ of f such that, for any point x of $\mathbb{R}^2$,

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Theorem (Béguin-Crovisier-Le Roux-Patou, M.)

Let f be a homeomorphism in $\mathrm{Homeo}_0(\mathbb{A})$. The homeomorphism f is a pseudo-rotation of the annulus of angle α if and only if f has conjugates arbitrarily close to the rotation R_α.

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Theorem (Kwapisz)

Let f be a homeomorphism in $\mathrm{Homeo}_0(\mathbb{T}^2)$. Let $\alpha \in \mathbb{R}^2/\mathbb{Z}^2$ be totally irrational. The homeomorphism f is a pseudo-rotation of the torus of angle α if and only if f has conjugates arbitrarily close to the rotation R_α.

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Is it still true without hypothesis on α?

Let S be a compact surface and $\tilde{S}$ be its universal cover. Let $D \subset \tilde{S}$ be a fundamental domain for the action of the group of deck transformations. The surface $\tilde{S}$ is endowed with a "natural" distance.

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Definition

A homeomorphism f in $\mathrm{Homeo}_0(S)$ is called a *pseudo-rotation* if

$$\lim_{n \to +\infty} \frac{\mathrm{diam}(\tilde{f}^n(D))}{n} = 0.$$

Let S be a compact surface with $\partial S \neq \emptyset$ and which is different from the annulus or the Möbius band.

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Is it still true when $\partial S = \emptyset$?

Definition

Let G be a finitely generated group and S be a finite generating set of G. An element g of G is *distorted* or a *distortion element* if

$$\lim_{n \to +\infty} \frac{l_S(g^n)}{n} = 0$$

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Proposition

Let $\varphi : G \to H$ be a group morphism. If $g \in G$ is distorted then $\varphi(g)$ is distorted.

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Proposition

Let $\varphi : G \to H$ be a group morphism. If $g \in G$ is distorted then $\varphi(g)$ is distorted.

Definition

Let G be any group. An element g of G is distorted if it belongs to a finitely generated subgroup of G in which it is distorted.

Lemma

Let S be a compact surface. If f is a distortion element in $\mathrm{Homeo}_0(S)$, then f is a pseudo-rotation.

Let M be a compact manifold.

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Proposition

Let $f \in \mathrm{Homeo}_0(M)$. If the homeomorphism f has conjugates arbitrarily close to a distortion element in $\mathrm{Homeo}_0(M)$, then f is distorted in $\mathrm{Homeo}_0(M)$.

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Let $f \in \mathrm{Homeo}_0(M)$. If the homeomorphism f has conjugates arbitrarily close to a distortion element in $\mathrm{Homeo}_0(M)$, then f is distorted in $\mathrm{Homeo}_0(M)$.

Corollary

Let S be a compact orientable surface with $\partial S \neq \emptyset$. An element of $\mathrm{Homeo}_0(S)$ is a distortion element if and only if it is a pseudo-rotation.

Theorem

Let S be a compact orientable surface and $f \in \mathrm{Homeo}_0(S)$. Suppose

$$\lim_{n \to +\infty} \frac{\mathrm{diam}(\tilde{f}^n(D))}{\frac{n}{log(n)}} = 0.$$

Then f is distorted in $\mathrm{Homeo}_0(S)$.