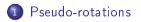
Conjugacy classes of pseudo-rotations and distortion elements in groups of homeomorphisms of surfaces

Emmanuel Militon

École Polytechnique

10 avril 2014



2 Distortion elements in groups of homeomorphisms of surfaces



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Lemma

Let $f \in Homeo_0(\mathbb{S}^1)$ with rotation number α . Then f has conjugates arbitrarily close to the rotation R_{α} .

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$$\mathbb{A} = \mathbb{R}/\mathbb{Z} \times [0, 1], \ \widetilde{\mathbb{A}} = \mathbb{R} \times [0, 1].$$

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A homeomorphism f in Homeo₀(\mathbb{A}) is called a *pseudo-rotation* if there exists α in \mathbb{R} and a lift $\tilde{f} : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ of f such that, for any point x of $\widetilde{\mathbb{A}}$,

$$\lim_{n\to+\infty}\frac{p_1(\tilde{f}^n(x))}{n}=\alpha,$$

where $p_1: ilde{\mathbb{A}} = \mathbb{R} imes [0,1] o \mathbb{R}$ is the projection.

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where $p_1 : \tilde{\mathbb{A}} = \mathbb{R} \times [0, 1] \to \mathbb{R}$ is the projection. The class of α in \mathbb{R}/\mathbb{Z} is the *angle* of the pseudo-rotation f.

$\mathbb{T}^2=\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z},\ \widetilde{\mathbb{T}^2}=\mathbb{R}^2.$

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A homeomorphism f in $\operatorname{Homeo}_0(\mathbb{T}^2)$ is called a *pseudo-rotation* if there exists α in \mathbb{R}^2 and a lift $\tilde{f} : \widetilde{\mathbb{T}^2} \to \widetilde{\mathbb{T}^2}$ of f such that, for any point x of \mathbb{R}^2 ,

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The class of α in \mathbb{T}^2 is the *angle* of the pseudo-rotation f.

Pseudo-rotations Distortion elements in groups of homeomorphisms of surface

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Theorem (Béguin-Crovisier-Le Roux-Patou, M.)

Let f be a homeomorphism in $Homeo_0(\mathbb{A})$. The homeomorphism f is a pseudo-rotation of the annulus of angle α if and only if f has conjugates arbitrarily close to the rotation R_{α} .

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Theorem (Kwapisz)

Let f be a homeomorphism in $\operatorname{Homeo}_0(\mathbb{T}^2)$. Let $\alpha \in \mathbb{R}^2/\mathbb{Z}^2$ be totally irrational. The homeomorphism f is a pseudo-rotation of the torus of angle α if and only if f has conjugates arbitrarily close to the rotation R_{α} .

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Is it still true without hypothesis on α ?

Let S be a compact surface and \tilde{S} be its universal cover. Let $D \subset \tilde{S}$ be a fundamental domain for the action of the group of deck transformations. The surface \tilde{S} is endowed with a "natural" distance.

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Definition

A homeomorphism f in Homeo₀(S) is called a *pseudo-rotation* if

$$\lim_{n\to+\infty}\frac{\operatorname{diam}(\tilde{f}^n(D))}{n}=0.$$

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Theorem (M.)

Let $f \in \operatorname{Homeo}_{0}(S)$.

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Let $f \in Homeo_0(S)$. The homeomorphism f is a pseudo-rotation if and only if it has conjugates arbitrarily close to the identity.

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Is it still true when $\partial S = \emptyset$?



2 Distortion elements in groups of homeomorphisms of surfaces

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Let G be a finitely generated group and S be a finite generating set of G. An element g of G is *distorted* or a *distortion element* if

$$\lim_{n\to+\infty}\frac{l_{\mathcal{S}}(g^n)}{n}=0$$

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Proposition

Let $\varphi : G \to H$ be a group morphism. If $g \in G$ is distorted then $\varphi(g)$ is distorted.

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Proposition

Let $\varphi : G \to H$ be a group morphism. If $g \in G$ is distorted then $\varphi(g)$ is distorted.

Definition

Let G be any group. An element g of G is distorted if it belongs to a finitely generated subgroup of G in which it is distorted.

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Lemma

Let S be a compact surface. If f is a distortion element in $Homeo_0(S)$, then f is a pseudo-rotation.

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Let M be a compact manifold.

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Let M be a compact manifold.

Proposition

Let $f \in \text{Homeo}_0(M)$. If the homeomorphism f has conjugates arbitrarily close to a distortion element in $\text{Homeo}_0(M)$, then f is distorted in $\text{Homeo}_0(M)$.

Let M be a compact manifold.

Proposition

Let $f \in \text{Homeo}_0(M)$. If the homeomorphism f has conjugates arbitrarily close to a distortion element in $\text{Homeo}_0(M)$, then f is distorted in $\text{Homeo}_0(M)$.

Corollary

Let S be a compact orientable surface with $\partial S \neq \emptyset$. An element of $Homeo_0(S)$ is a distortion element if and only if it is a pseudo-rotation.

Theorem

Let S be a compact orientable surface and $f \in \operatorname{Homeo}_0(S)$. Suppose

$$\lim_{n \to +\infty} \frac{\operatorname{diam}(\tilde{f}^n(D))}{\frac{n}{\log(n)}} = 0.$$

Then f is distorted in Homeo₀(S).

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